Probabilistic Aspects of Fatigue

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Probabilistic Aspects of Fatigue

- Introduction
- Statistical Techniques
- Sources of Variability

Probabilistic Models

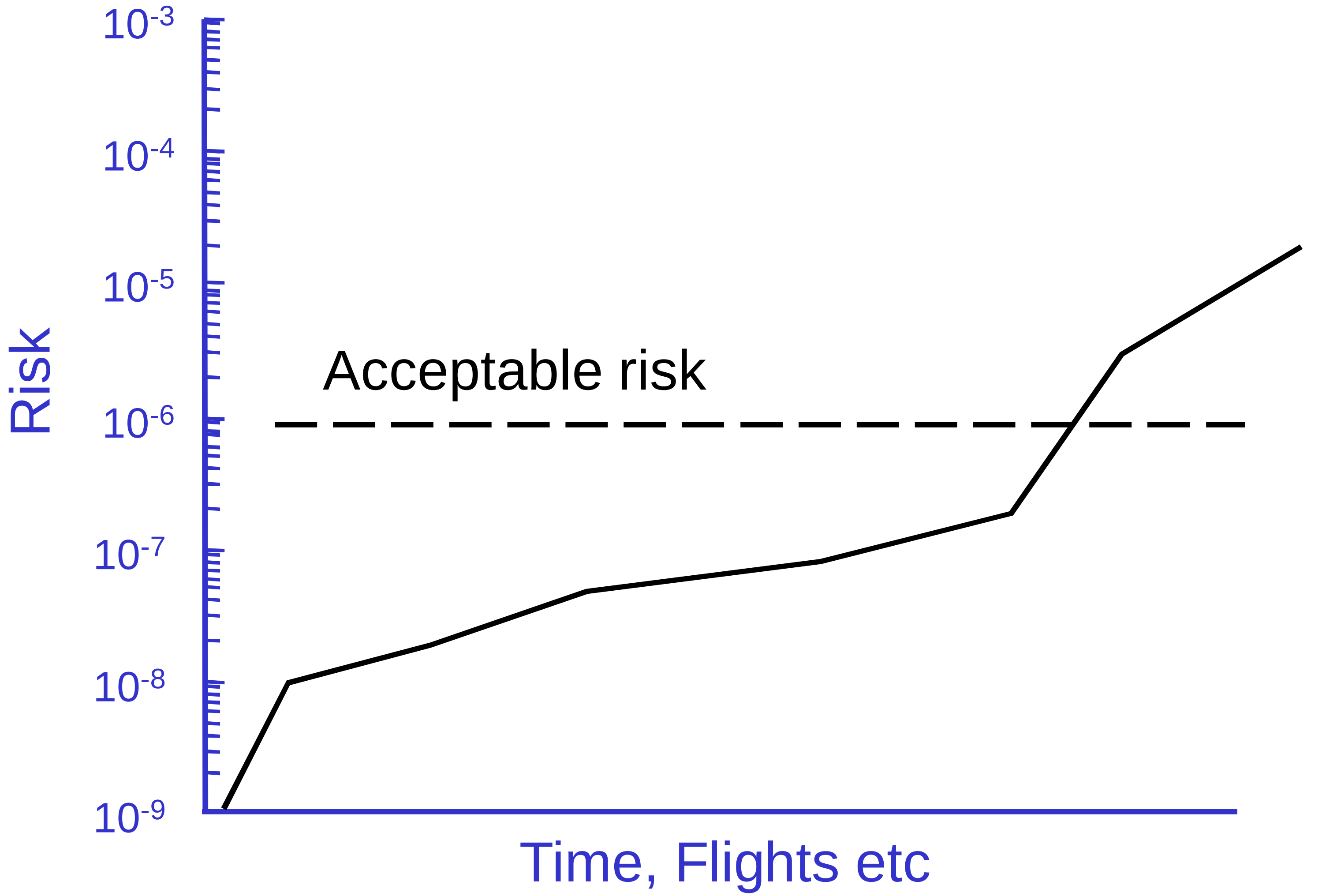
- Probabilistic models are no better than the underlying deterministic models
- They require more work to implement
- Why use them?

Quality and Cost

- Taguchi
  - Identify factors that influence performance
  - Robust design – reduce sensitivity to noise
  - Assess economic impact of variation

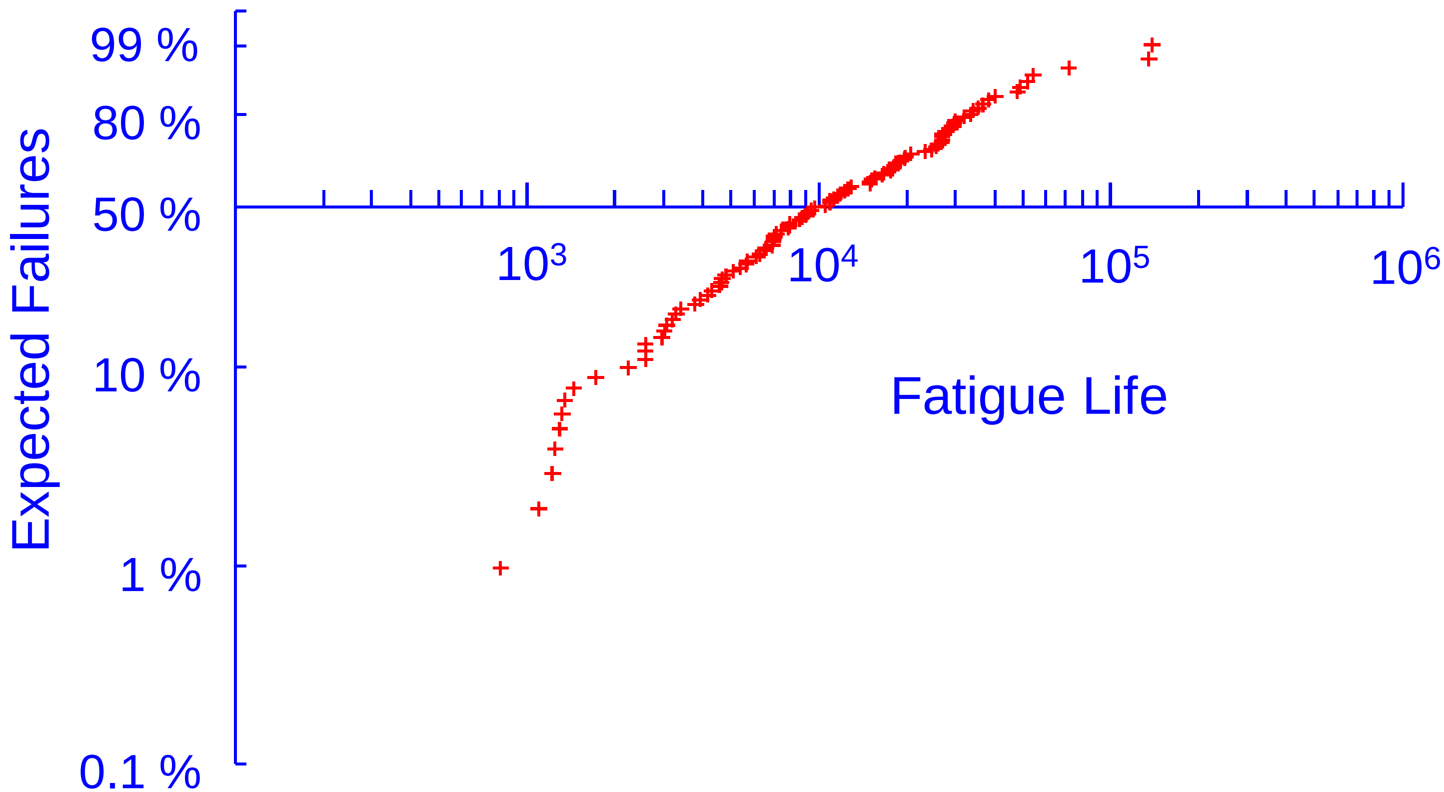
- Risk / Reliability
  - What is the increased risk from reduced testing ?

Risk

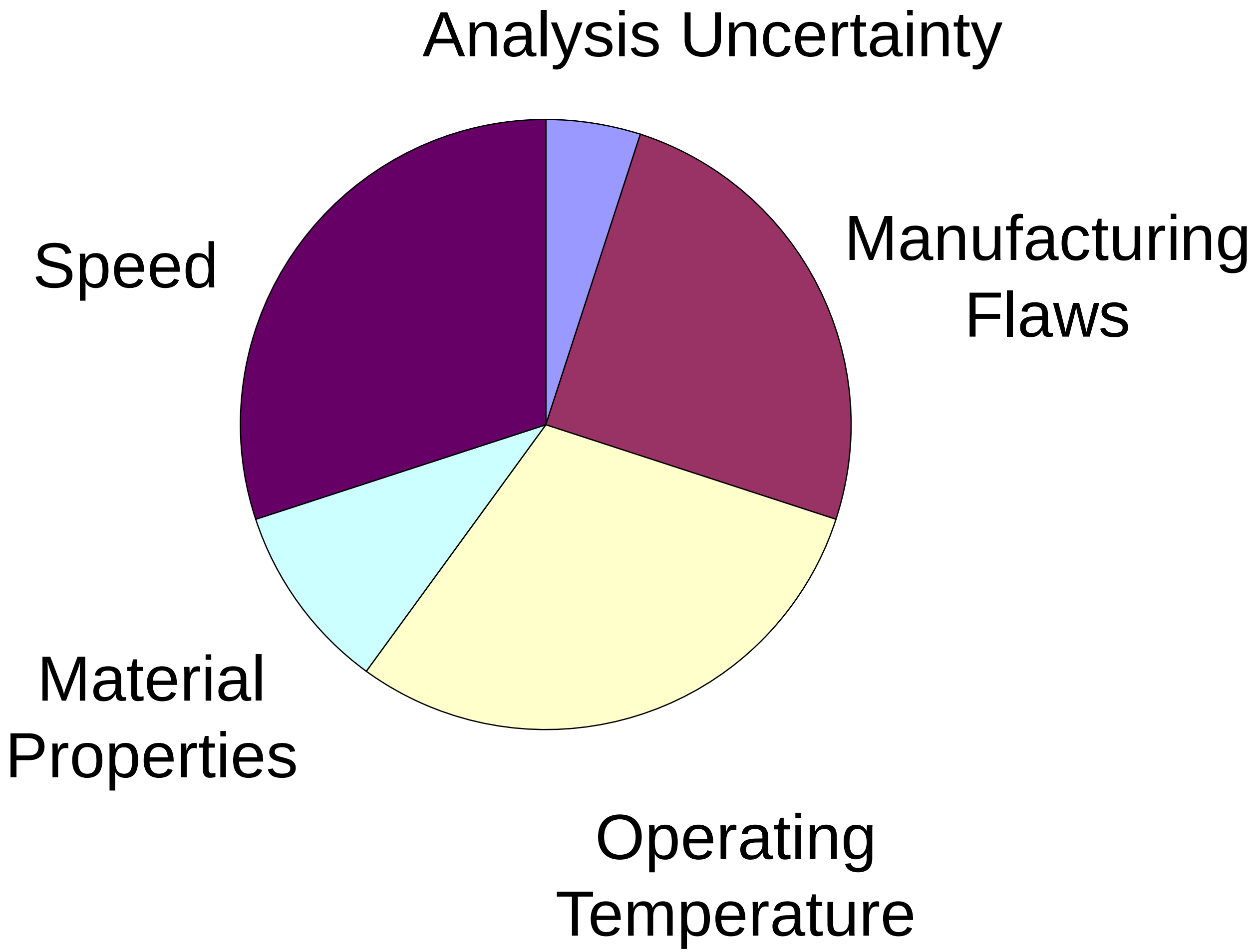


10-3
10-4
10-5
10-6
10-7
10-8
10-9
Risk
Acceptable risk
Time, Flights etc

Reliability



Risk Contribution Factors

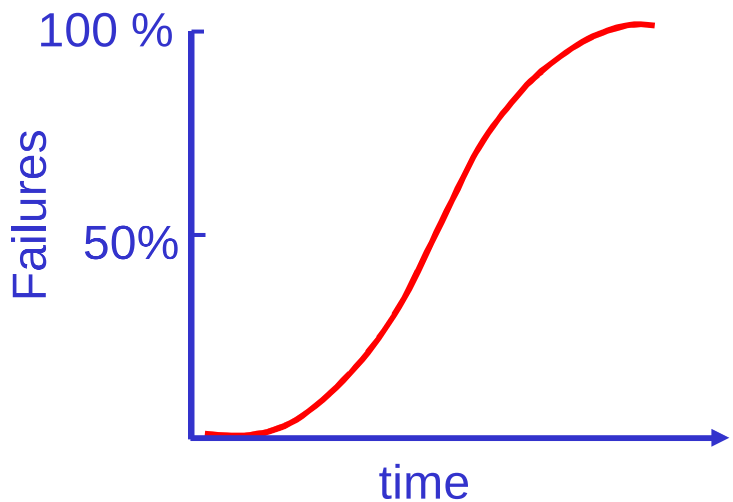


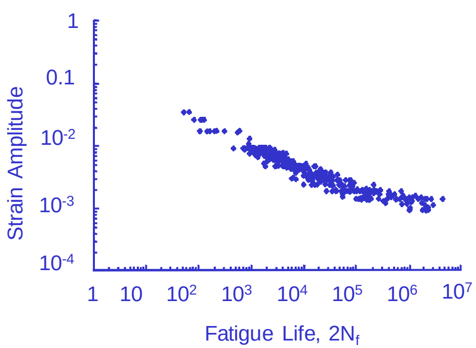
Uncertainty and Variability

customers

← Stress →

usage





materials

← Strength →

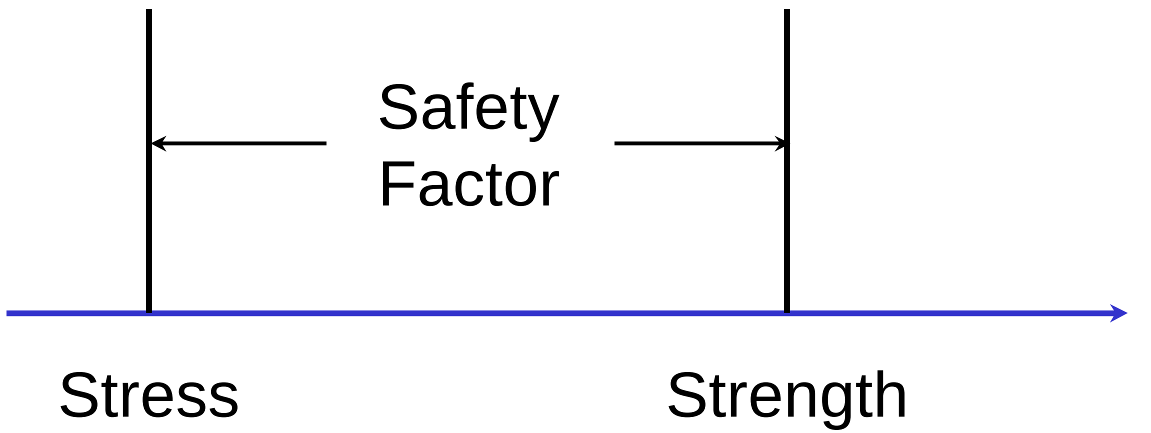
manufacturing

Deterministic versus Random

Deterministic – from past measurements the future position of a satellite can be predicted with reasonable accuracy

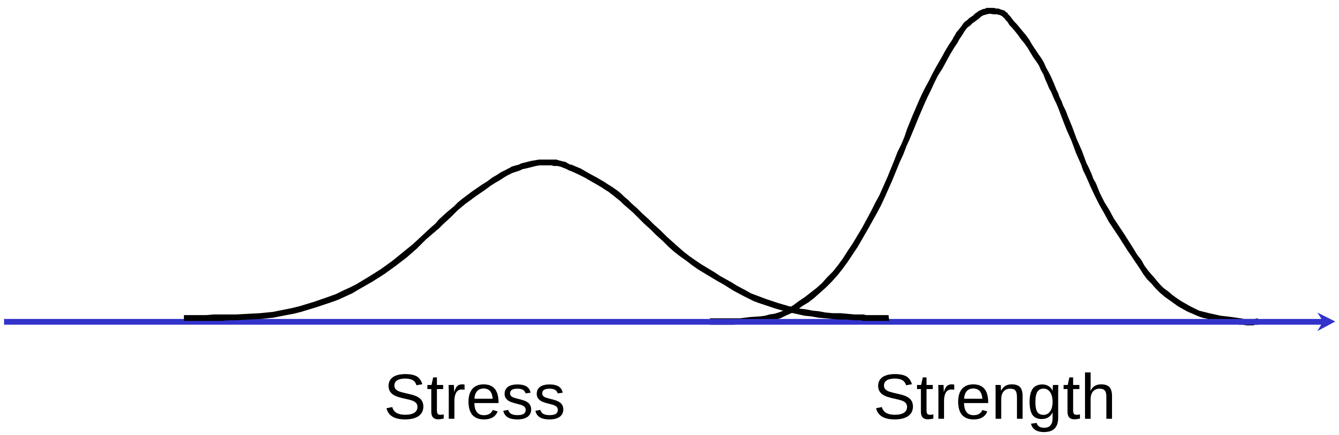
Random – from past measurements the future position of a car can only be described in terms of probability and statistical averages

Deterministic Design



Variability and uncertainty is accommodated by introducing safety factors. Larger safety factors are better, but how much better and at what cost?

Probabilistic Design



Reliability = 1 – P(Stress > Strength)

3σ Approach

3σ contains 99.87% of the data

$$P(s < S) = 2.3\ 10^{-3}$$

If we use 3σ on both stress and strength

$$P(\text{failure}) = P(\Sigma \geq s \cap s \leq S) = 5.3\ 10^{-6} \approx 4.5\sigma$$

The probability of the part with the lowest strength having the highest stress is very small

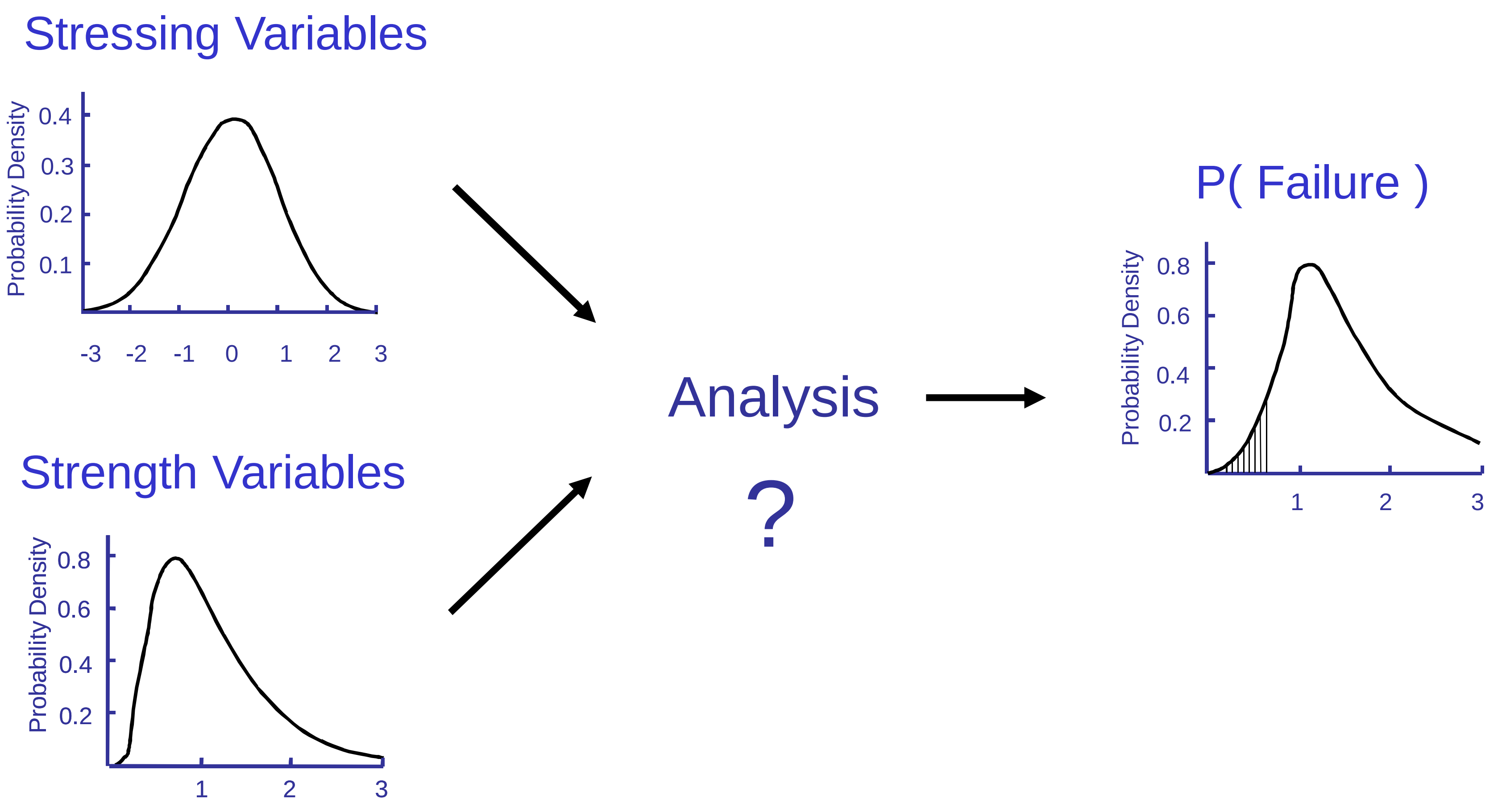
For 3 variables, each at 3 σ:

$$P(\text{failure}) = 1.2\ 10^{-8} \approx 5.7\sigma$$

Benefits

- Reduces conservatism (cost) compared to assuming the “worst case” for every design variable
- Quantifies life drivers – what are the most important variables and how well are they known or controlled ?
- Quantifies risk

Reliability Analysis



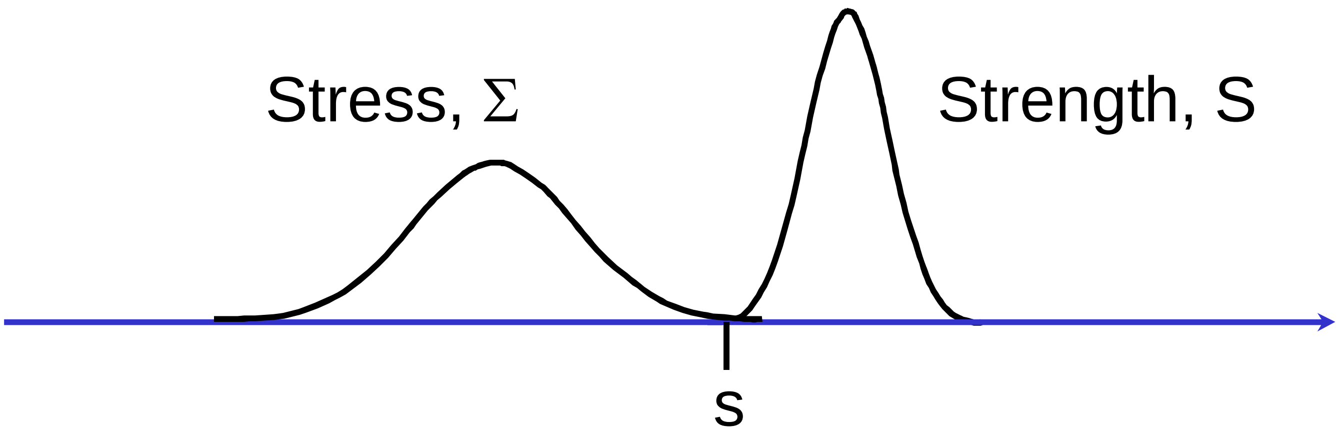
Probabilistic Analysis Methods

- Monte Carlo
  - Simple
  - Hypercube sampling
  - Importance sampling
- Analytical
  - First order reliability method FORM
  - Second order reliability method SORM

Statistical Techniques

- Normal Distributions
- LogNormal Distributions
- Monte Carlo
- Distribution Fitting

Failure Probability



Let Σ be the stress and S the fatigue strength

Given the distributions of Σ and S find the probability of failure

$$P(\Sigma \geq s \cap s \leq S)$$

Normal Variables

Linear Response Function

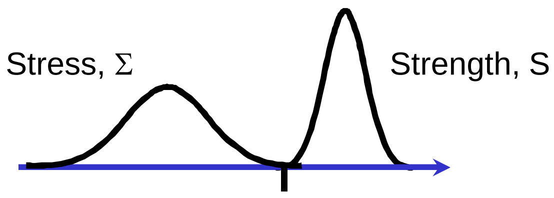
$$Z = a_o + \sum_{i=1}^{n} a_i X_i$$

$$X_i \sim N(\mu_i , C_i)$$

$$\mu_z = a_o + \sum_{i=1}^{n} a_i \mu_i$$

$$\sigma_z = \sqrt{\sum_{i=1}^{n} a_i^2 \sigma_i^2}$$

Calculations



$S \sim N(200, 0.1)\ \sigma_S = 20$

$\Sigma \sim N(100, 0.2)\ \sigma_\Sigma = 20$

Safety factor of 2

Let Z be a random variable:

$$Z = S - \Sigma$$

$$\mu_Z = \mu_S - \mu_\Sigma$$

$$\mu_Z = 200 - 100 = 100$$

$$\sigma_Z = \sqrt{\sigma_S^2 + \sigma_\Sigma^2}$$

$$\sigma_Z = \sqrt{20^2 + 20^2} = 28.2$$

Failure Probability

$$Z = S - \Sigma$$

Failure will occur whenever Z <= 0

$$Z = \mu_z - z\,\sigma_z = 0$$

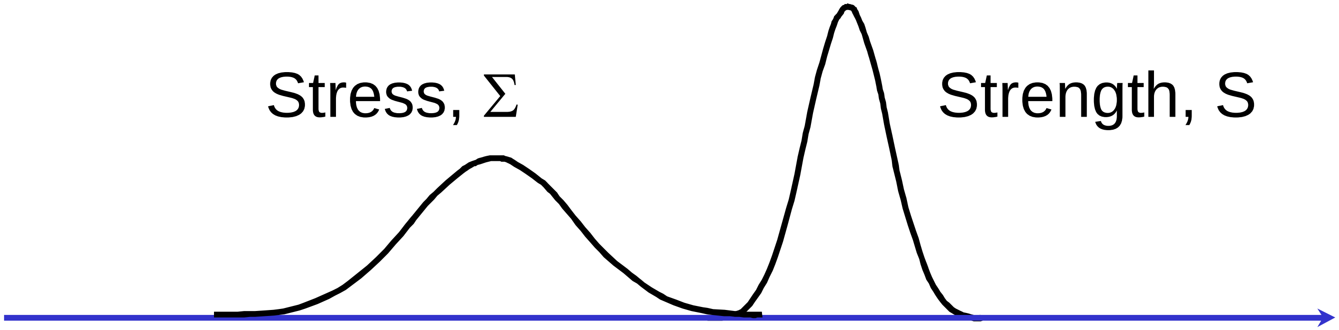
$$z = \frac{\mu_Z}{\sigma_Z} = \frac{100}{28.2}$$

z = 3.54 standard deviations

$$P(\text{failure}) = 2 \times 10^{-4}$$

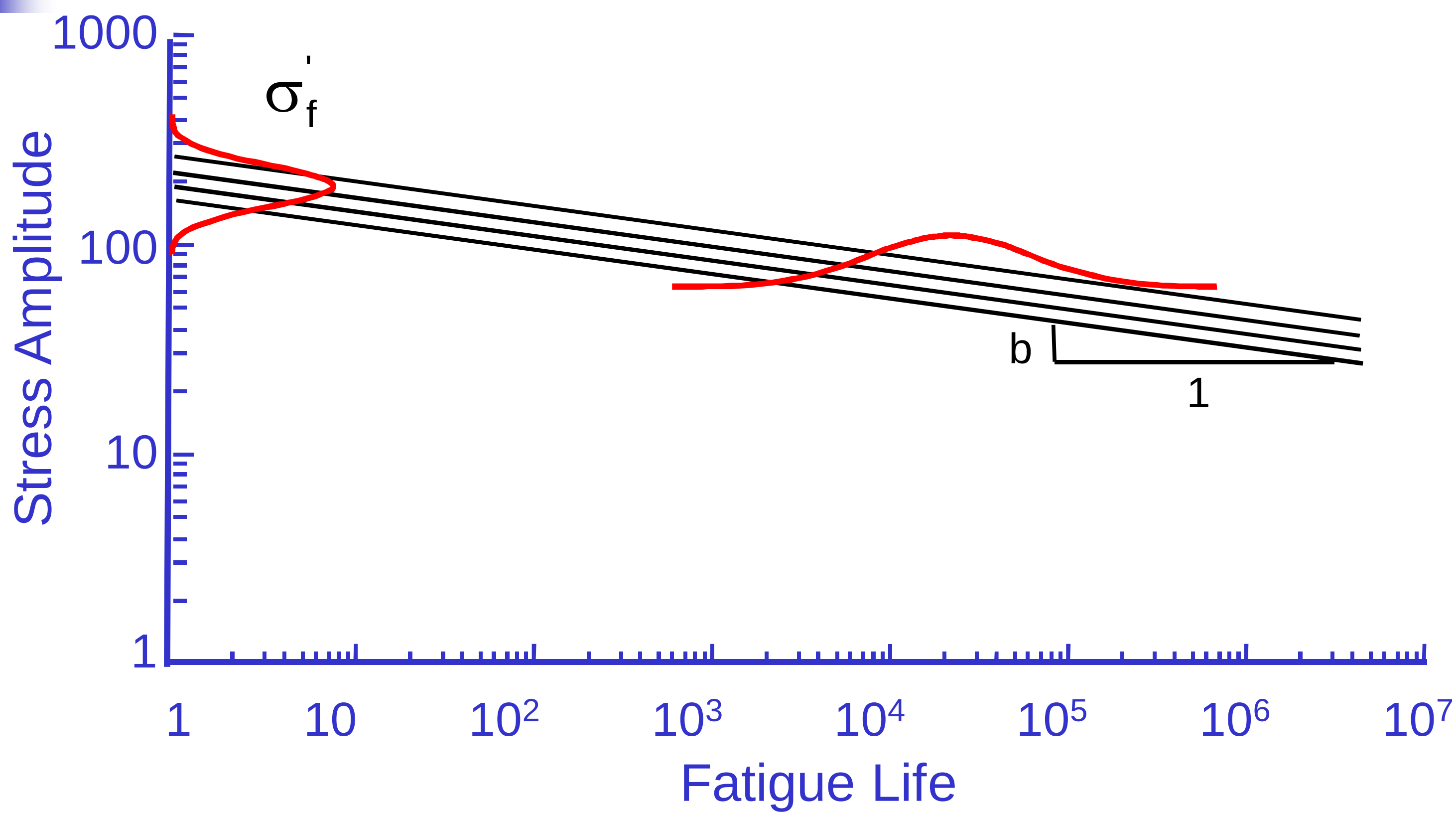
For this case only, a safety factor of 2 means a probability of failure of 2×10^{-4}. Other situations will require different safety factors to achieve the same reliability.

Failure Distribution



What is the expected distribution in fatigue lives?

Fatigue Data



$$\sigma_f' = \frac{\Delta S}{2(2N_f)^b}$$

$$2N_f = \left(\frac{\Delta S}{2\sigma_f'}\right)^{\frac{1}{b}}$$

LogNormal Variables

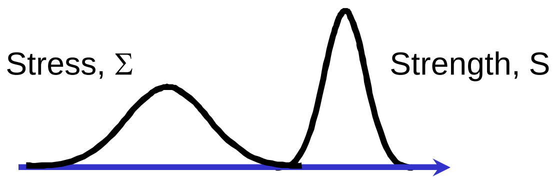
$$Z = a_o \prod_{i=1}^{n} X_i^{a_i}$$

a's are constant and $X_i \sim LN(\, x_i \,,\, C_i \,)$

median $\quad \overline{Z} = a_o \prod_{i=1}^{n} \overline{X}_i^{a_i}$

COV $\quad C_Z = \sqrt{\prod_{i=1}^{n} \left(1 + C_{X_i}^{\ 2}\right)^{a_i^2} - 1}$

Calculations



$$2N_f = \left(\frac{\Delta S}{2\sigma_f'}\right)^{\frac{1}{b}}$$

$\sigma_f' \sim LN(\,1000\ ,\ 0.1\,)\ \sigma = 100$

$\frac{\Delta S}{2} \sim LN(\,250\ ,\ 0.2\,)\ \sigma = 50$

$b = -0.125$

$$Z = 2N_f = \left(\frac{\Delta S}{2}\right)^{-8} \sigma_f'^{\,8}$$

$$\overline{Z} = 2N_f = \left(\frac{\Delta \overline{S}}{2}\right)^{-8} \overline{\sigma}_f'^{\,8}$$

$$COV_Z = \sqrt{\left(1 + COV_{\Delta S}{}^2\right)^{-8^2} \left(1 + COV_{\sigma_f}{}^2\right)^{8^2} - 1}$$

Results

	$\Delta S/2$	σ'_f	$2N_f$		Percentile	Life
μ_x	250	1000	355,368		99.9	17,706,069
COV_x	0.2	0.1	4.72		99	4,566,613
					95	1,363,200
$\mu_{\ln x}$	5.50	6.90	11.21		90	715,589
X	245	995	73,676		50	73,676
σ_x	50	100	1,676,831		10	7,586
$\sigma_{\ln x}$	0.198	0.100	1.774		5	3,982
					1	1,189
b =	-0.125				0.1	307

Monte Carlo Methods

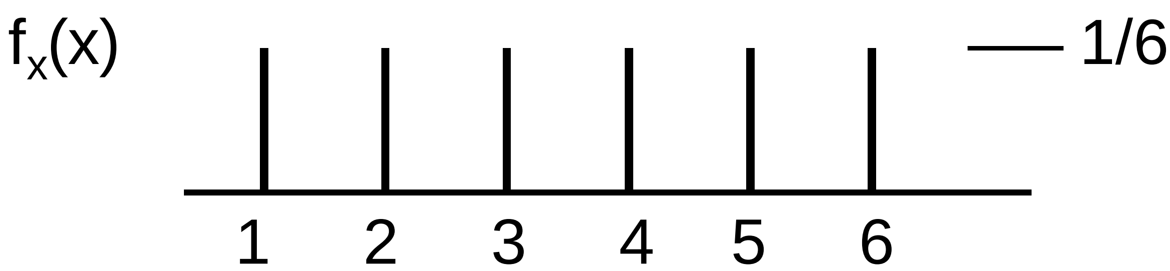
$$\frac{K_f \Delta S}{2} = \sqrt{E\left(\frac{\sigma_f^{'2}}{E}\left(2N_f\right)^{2b} + \sigma_f^{'}\varepsilon_f^{'}\left(2N_f\right)^{b+c}\right)}$$

Given random variables for K_f, ΔS, $\sigma_f^{'}$ and $\varepsilon_f^{'}$
Find the distribution of $2N_f$

$Z = 2N_f = ?$

Simple Example

Probability of rolling a 3 on a die



Uniform discrete distribution

Computer Simulation

1. Generate n random numbers between 1 and 6, all integers

2. Count the number of 3's

Let X_i = 1 if 3
0 otherwise

$$P(3) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

EXCEL

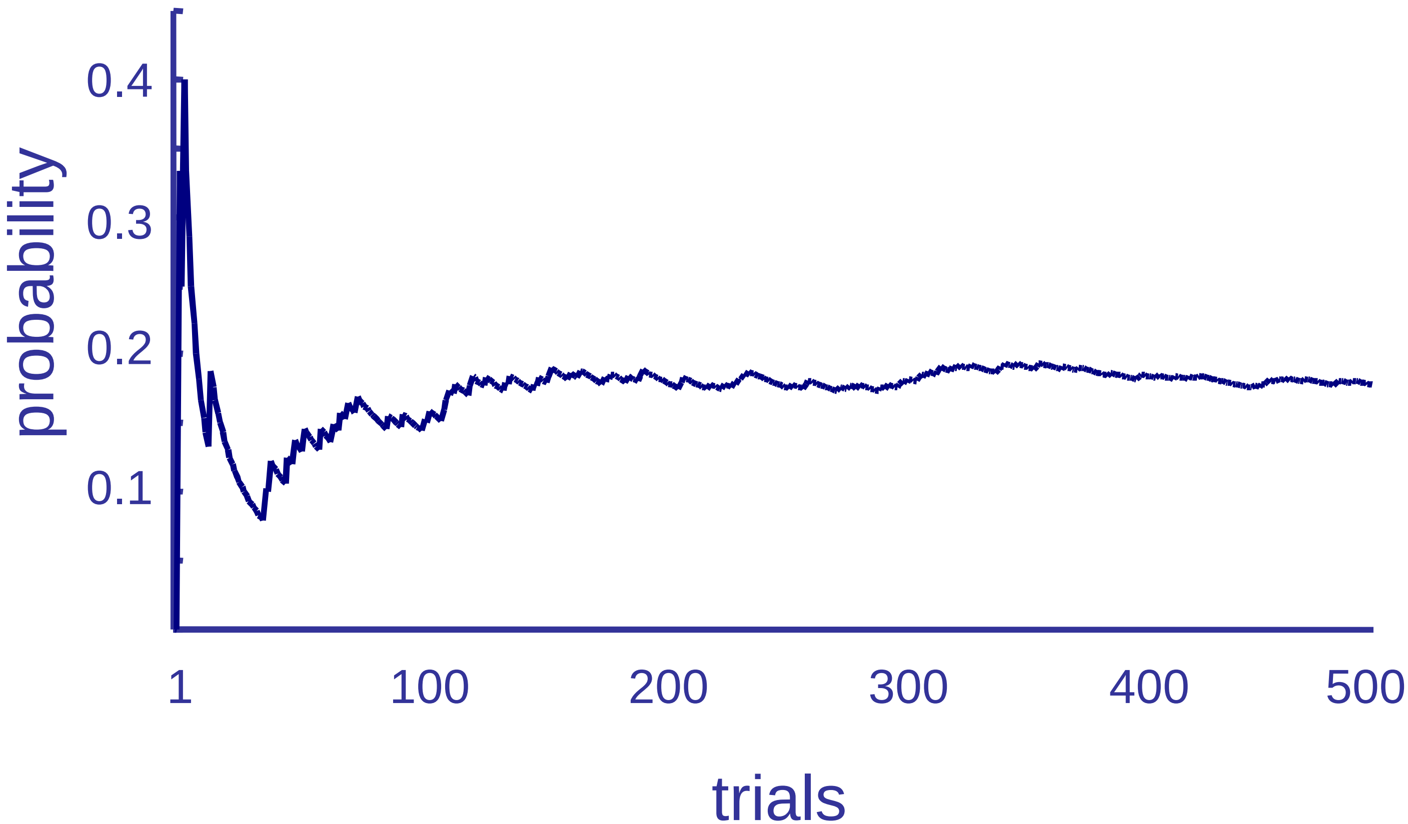
=ROUNDUP(6 * RAND() , 0)

=IF(A1 = 3 , 1 , 0)

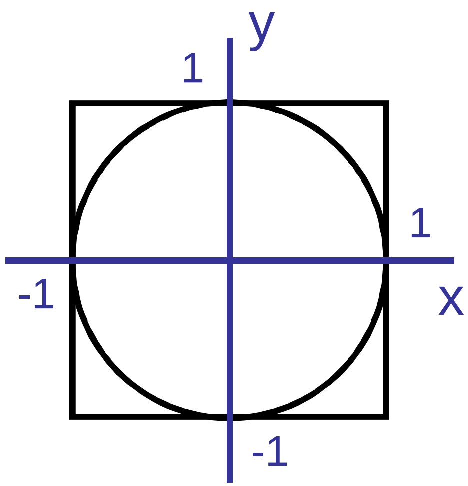
=SUM(B1:B1)/ROW(B1)

5	0	0
3	1	0.5
4	0	0.333333
4	0	0.25
5	0	0.2
6	0	0.166667
1	0	0.142857
3	1	0.25
3	1	0.333333
6	0	0.3

Results



Evaluate π



P(inside circle)

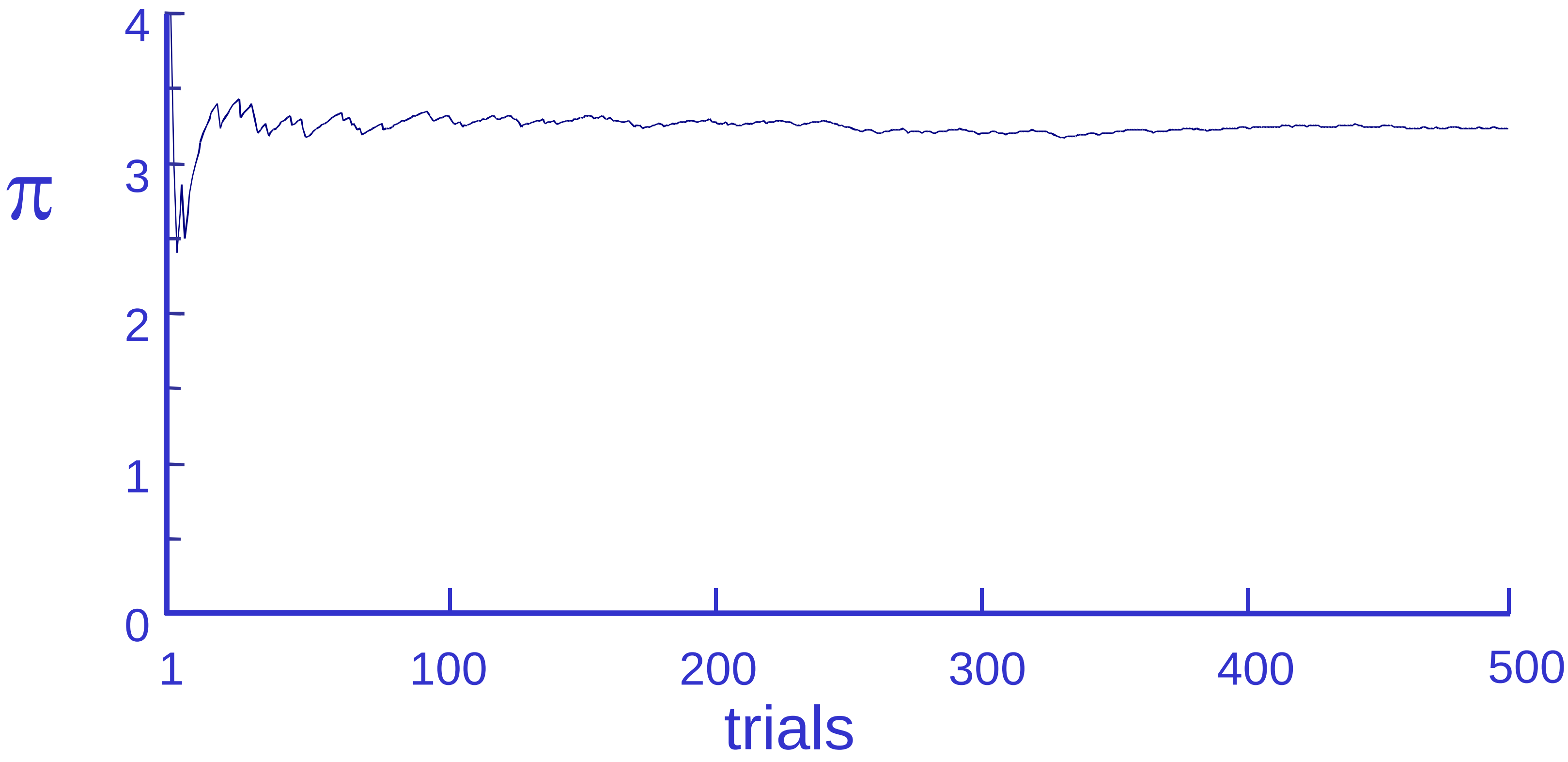
$$P = \frac{\pi r^2}{4}$$

$$\pi = 4\ P$$

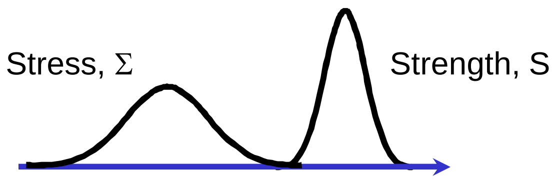
x = 2 * RAND() - 1
y = 2 * RAND() - 1

IF($x^2 + y^2 < 1$, 1 , 0)

π



Monte Carlo Simulation



$$2N_f = \left(\frac{\Delta S}{2\sigma_f'}\right)^{\frac{1}{b}}$$

σ_f' ~ LN(1000 , 0.1) σ = 100

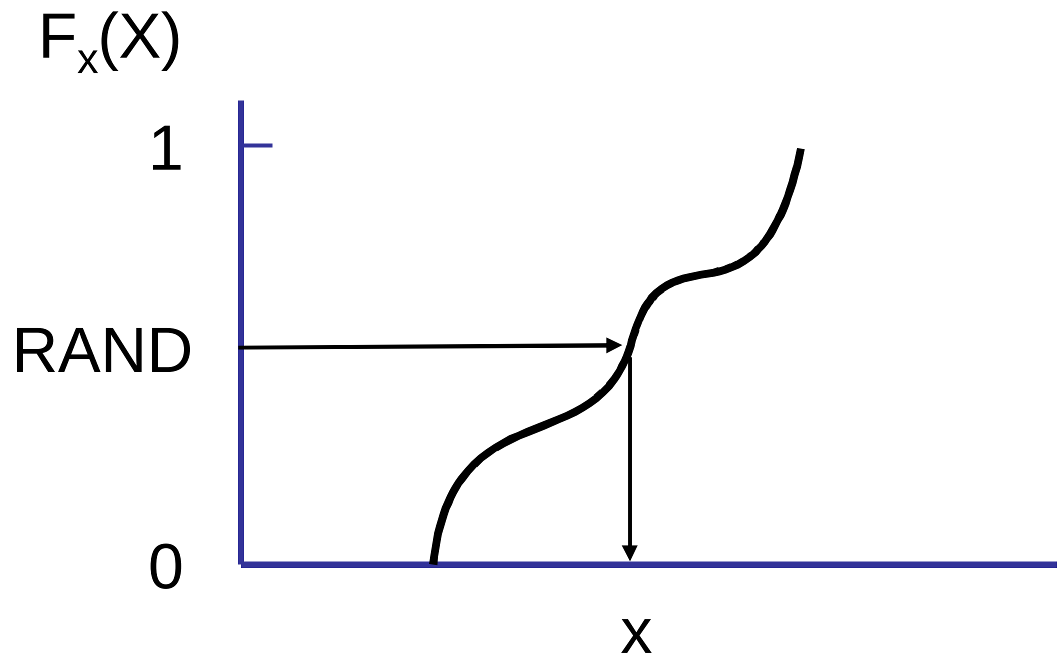
$\frac{\Delta S}{2}$ ~ LN(250 , 0.2) σ = 50

b = -0.125

Randomly choose values of S and σ_f' from their distributions

Repeat many times

Generating Distributions



Randomly choose a value between 0 and 1

$$x = F_x^{-1}(\ RAND\ )$$

Generating Distributions in EXCEL

Normal

=NORMINV(RAND(),μ,σ)

Log Normal

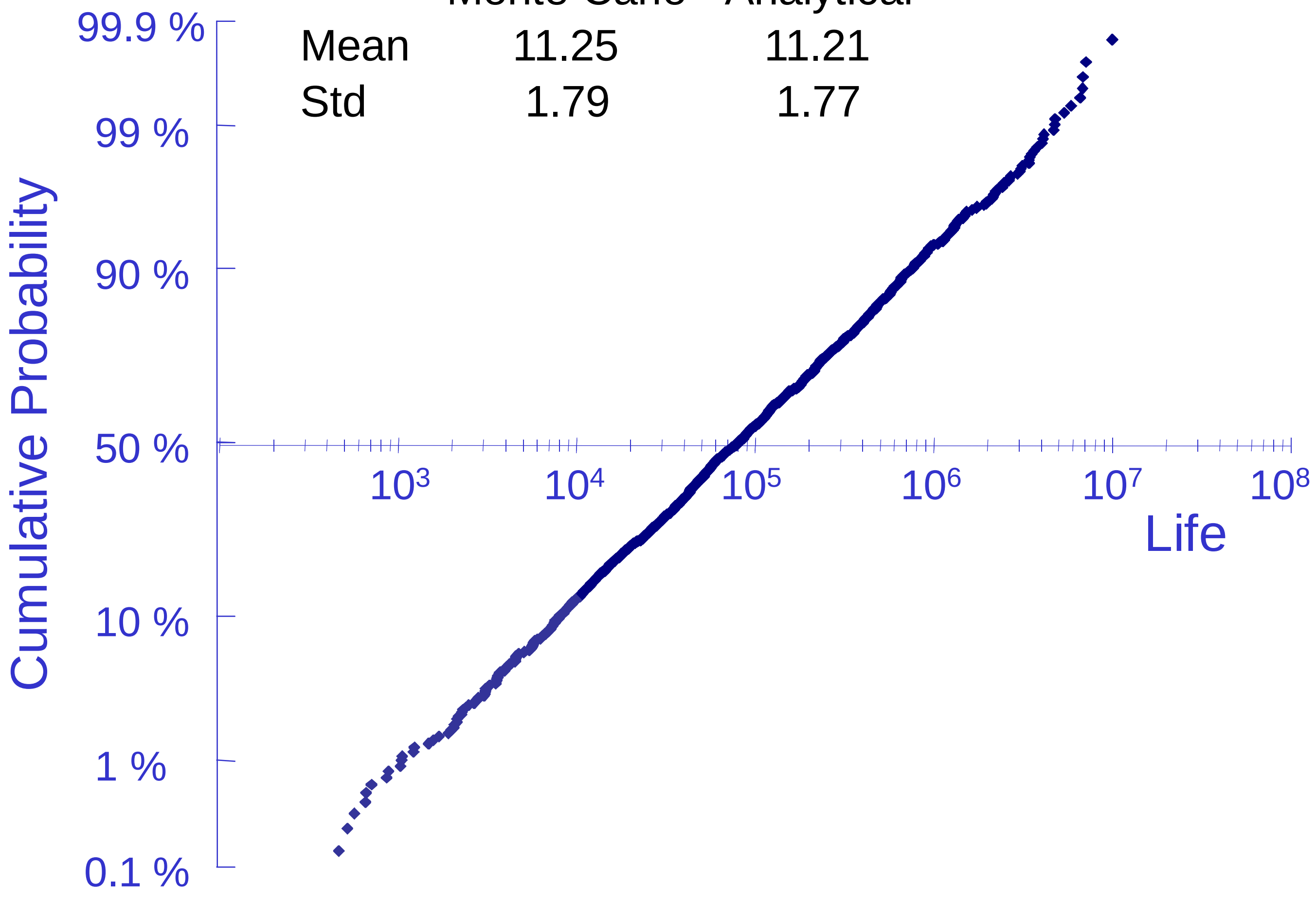
=LOGINV(RAND(),lnμ,lnσ)

EXCEL

σ_f'	$\frac{\Delta S}{2}$	$2N_f$
893	204	134,677
1102	301	32,180
852	285	6,355
963	173	929,249
1050	283	35,565
1080	265	77,057
965	313	8,227
1073	213	420,456
1052	226	224,000
954	322	5,878
965	240	68,671
993	207	277,192
1191	368	11,967
831	210	59,473

Simulation Results

	Monte Carlo	Analytical
Mean	11.25	11.21
Std	1.79	1.77

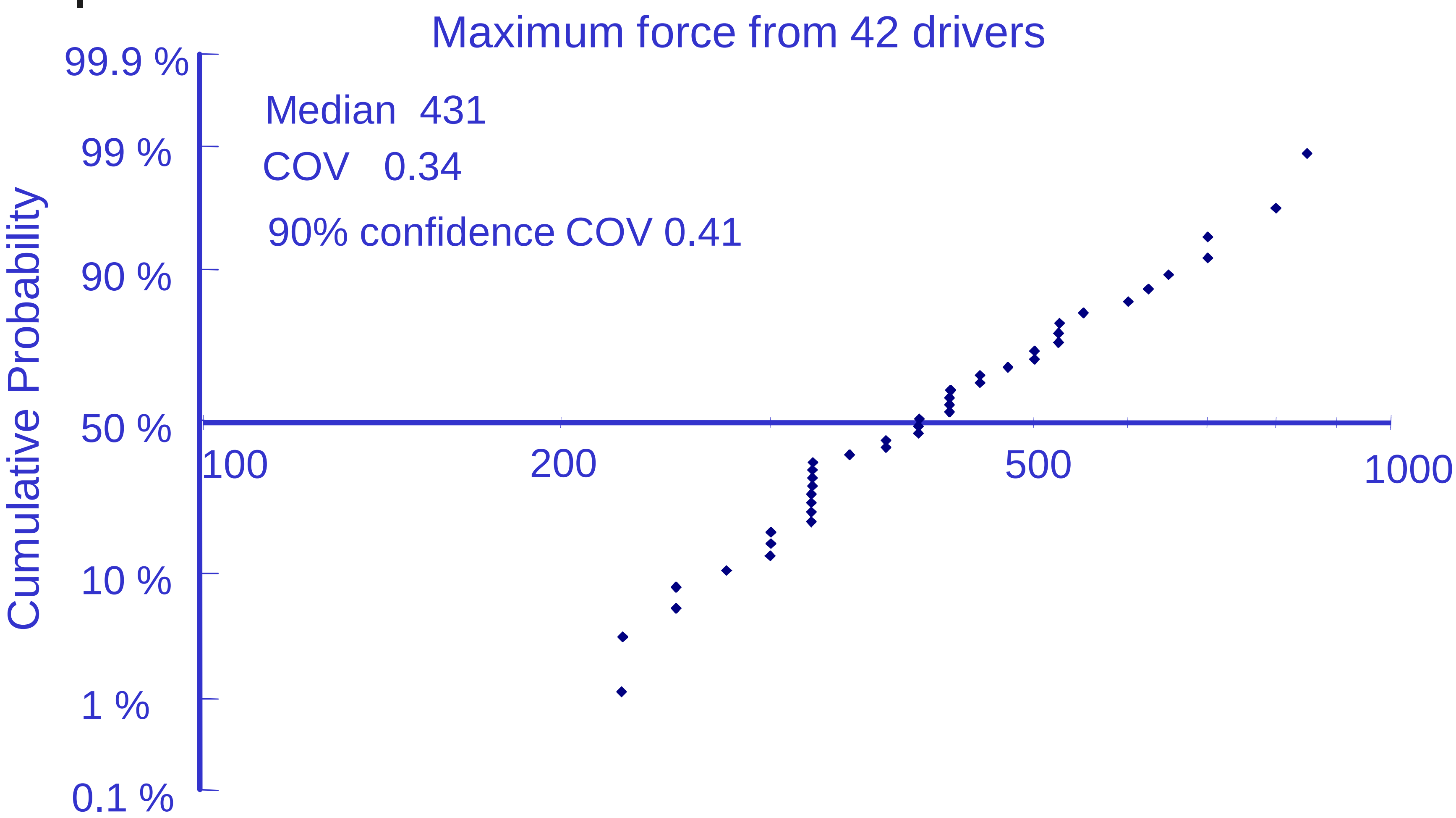


Summary

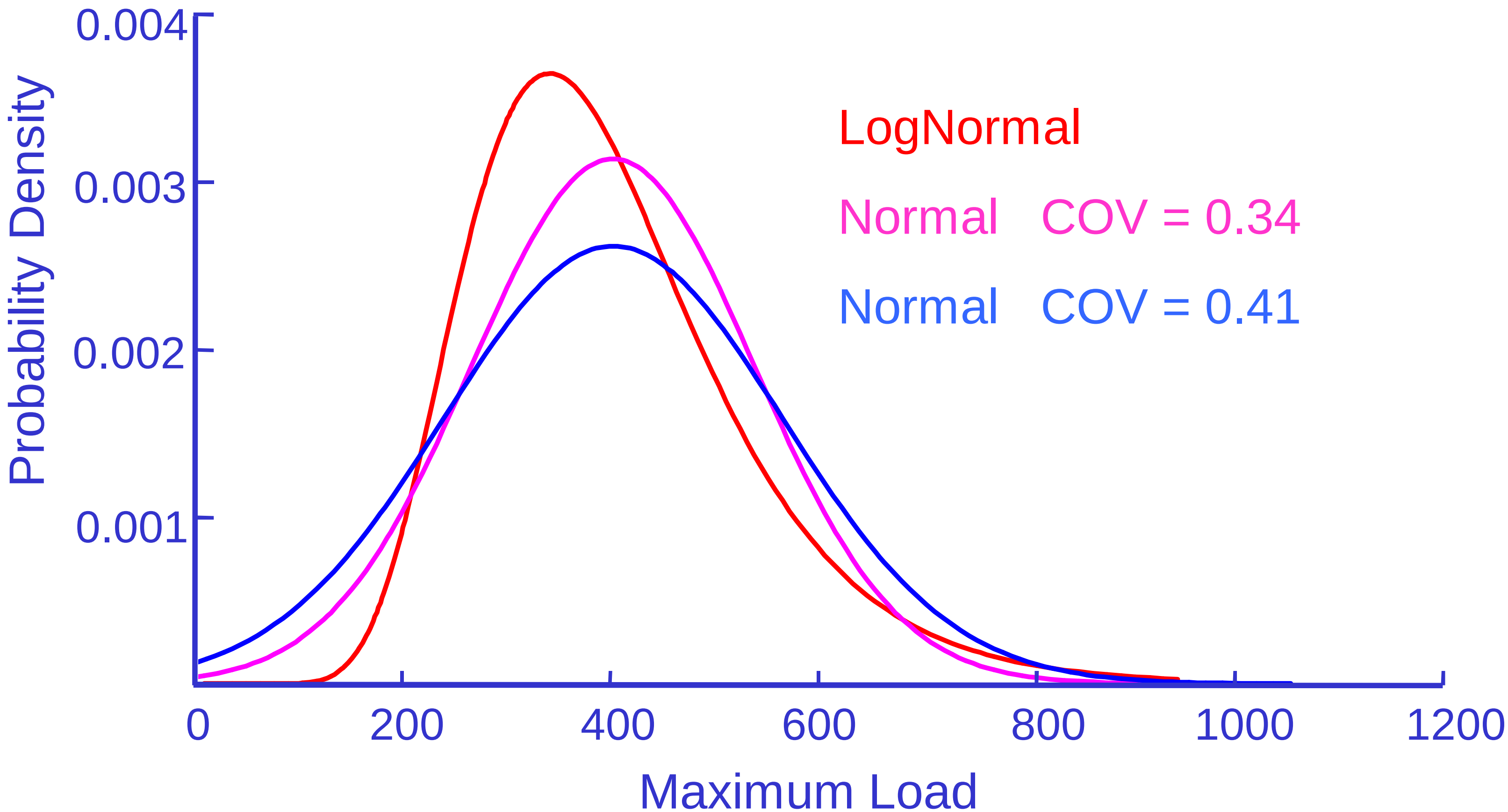
Simulation is relatively straightforward and simple

Obtaining the necessary input data and distributions is difficult

Maximum Load Data

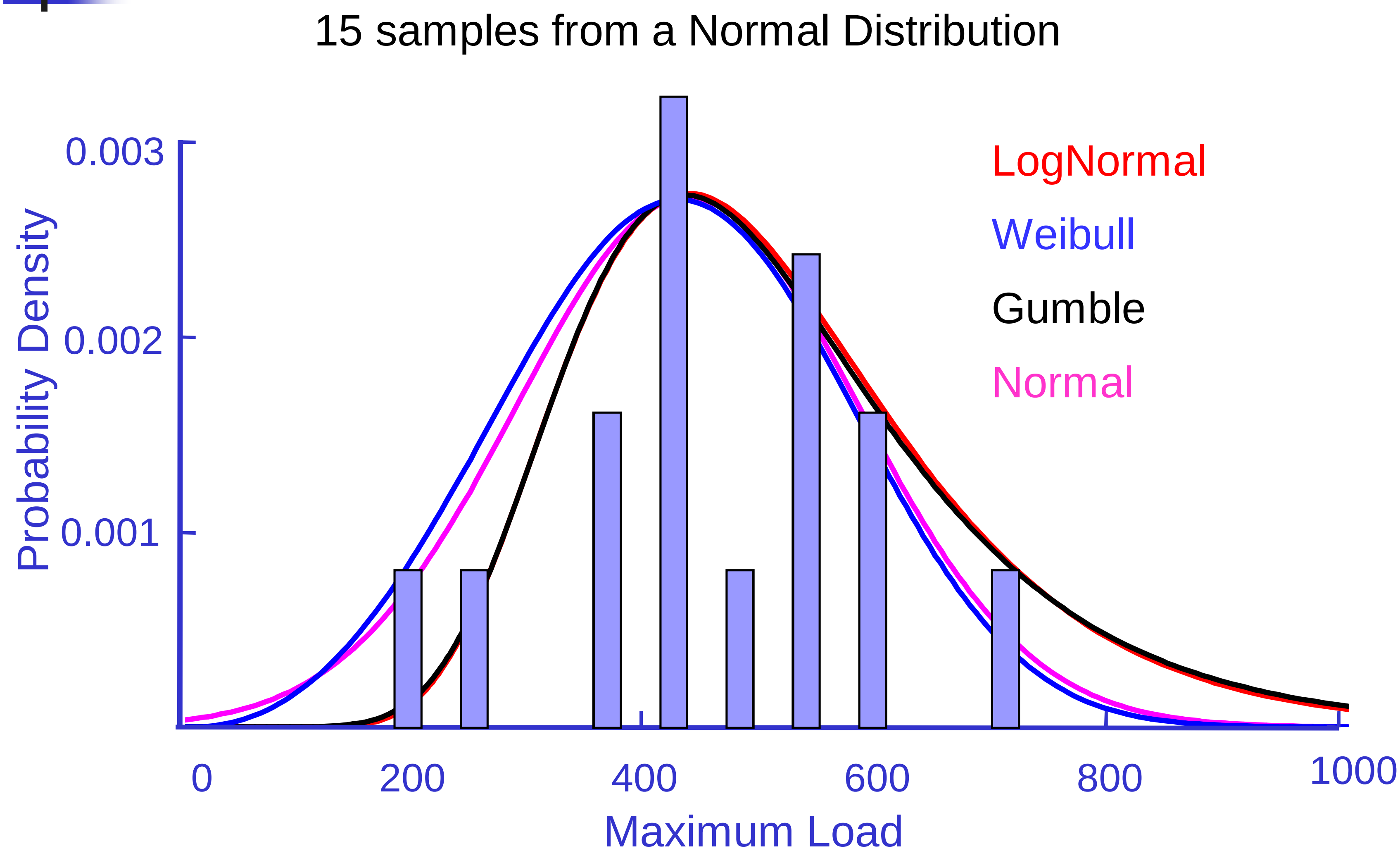


Maximum Load Data



Uncertainty in Variance is just as important, perhaps more important than the choice of the distribution

Choose the “Best” Distribution



Distributions

- Normal
  - Strength
  - Dimensions
- LogNormal
  - Fatigue Lives
  - Large variance in properties or loads
- Gumble
  - Maximums in a population
- Weibull
  - Fatigue Lives

Central Limit Theorem

If X_1, X_2, X_3 ….. X_n is a random sample from the population, with sample mean $\overline{X}$, then the limiting form of

$$Z = \frac{\overline{X} - \mu_X}{\sigma / \sqrt{n}}$$

as $n \to \infty$ is the standard normal distribution

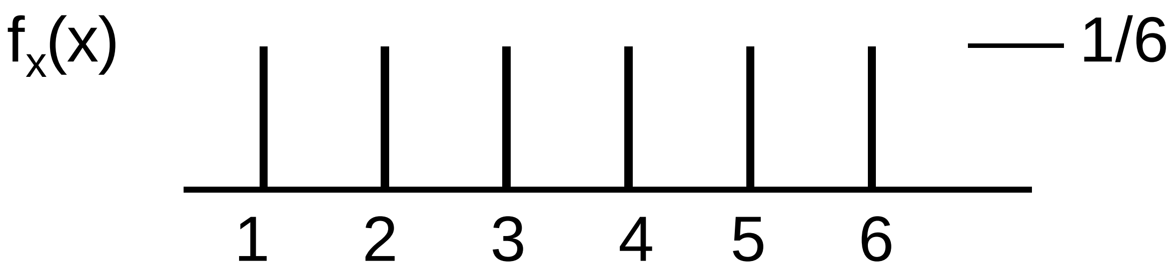
Translation

When there are many variables affecting the outcome,
The final result will be normally distributed even if the individual variable distributions are not.

As a result, normal distributions are frequently assumed for all of the input variables

Example

Probability of rolling a die

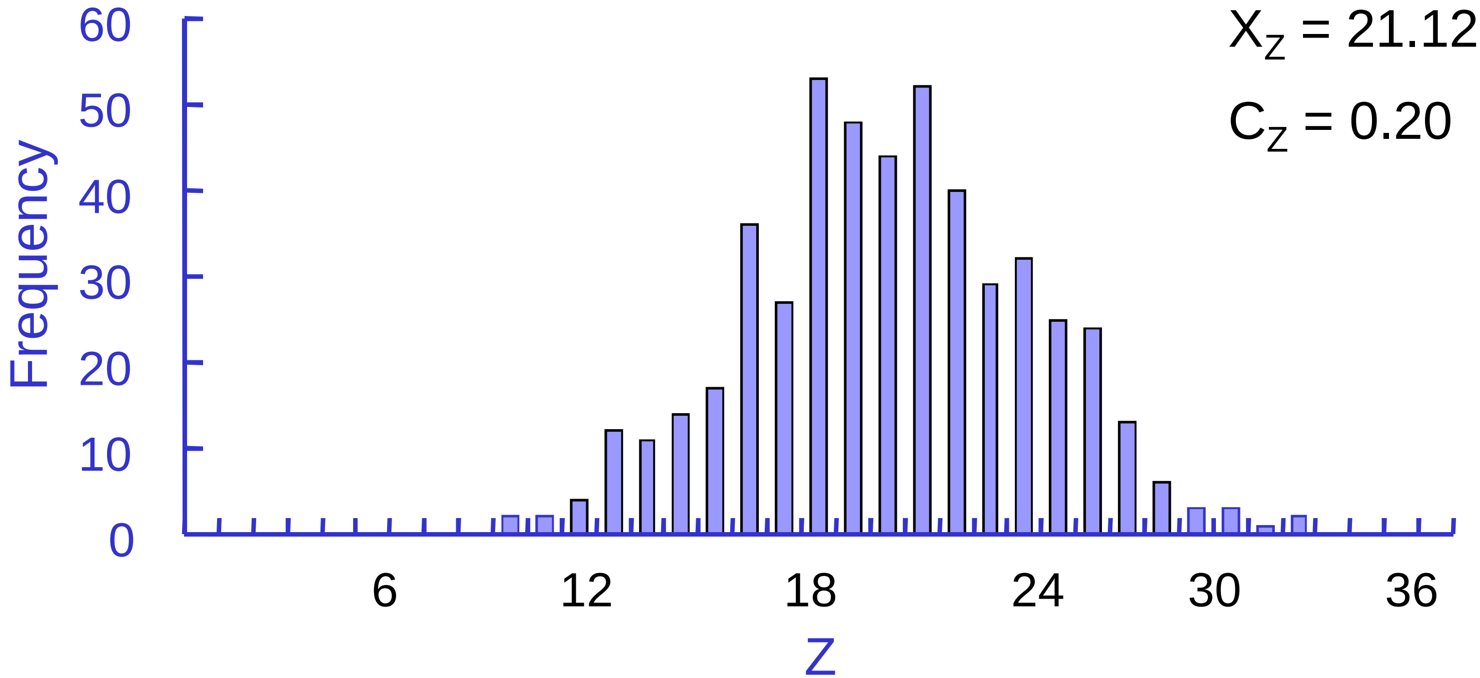


Uniform discrete distribution

Let Z be the summation of six dice

$$Z = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$$

Results



500 trials

$\bar{X}_Z = 21.12$

$C_Z = 0.20$

Central limit theorem states that the result should be normal for large n

Central Limit Theorem

Sums: $Z = X_1 \pm X_2 \pm X_3 \pm X_4 \pm \ldots\ldots X_n$

$Z \rightarrow$ Normal as n increases

Products: $Z = X_1 \bullet X_2 \bullet X_3 \bullet X_4 \bullet \ldots\ldots X_n$

$Z \rightarrow$ LogNormal as n increases

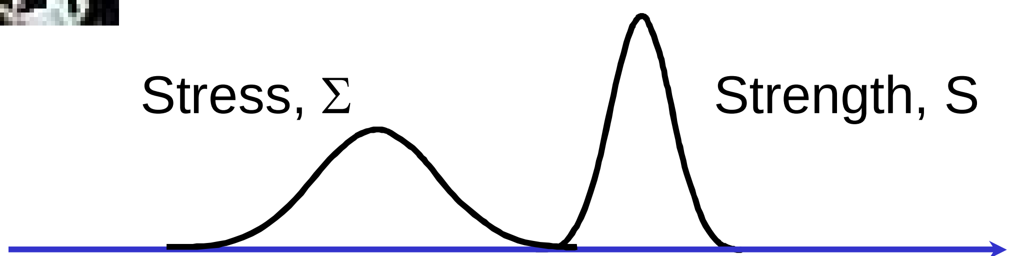
Normal and LogNormal distributions are often employed for analysis even though the underlying population distribution is unknown.

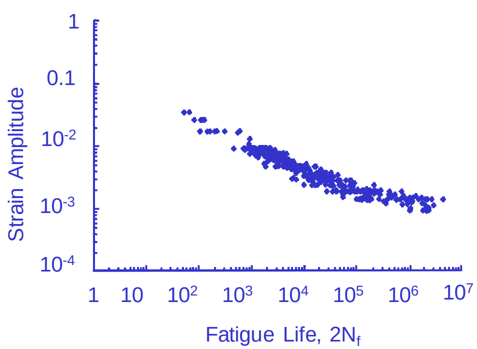
Key Points

- All variables are random and can be characterized by a statistical distribution with a mean and variance.
- The final result will be normally distributed even if the individual variable distributions are not.

Sources of Variability

customers ← Stress → usage





materials ← Strength → manufacturing

Variability and Uncertainty

Variability: Every apple on a tree has a different mass.

Uncertainty: The variety of the apple is unknown.

Variability: Fracture toughness of a material

Uncertainty: The correct stress intensity factor solution

Sources of Variability

- Stress Variables
  - Loading
  - Customer Usage
  - Environment
- Strength Variables
  - Material
  - Processing
  - Manufacturing Tolerance
  - Environment

Sources of Uncertainty

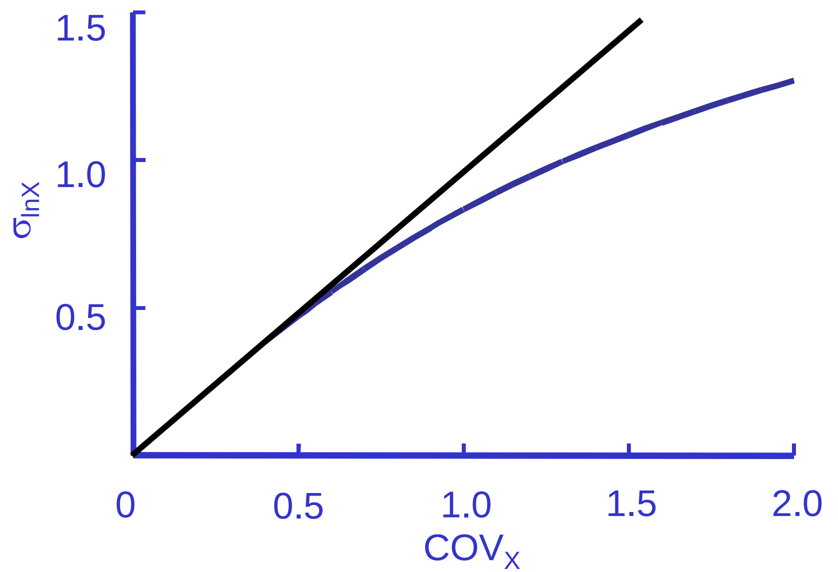
- Statistical Uncertainty
  - Incomplete data (small sample sizes)
- Modeling Error
  - Analysis assumptions
- Human Error
  - Calculation errors
  - Judgment errors

Modeling Variability

Central Limit Theorem:

Products: $Z = X_1 \bullet X_2 \bullet X_3 \bullet X_4 \bullet \ldots\ldots X_n$

$Z \rightarrow$ LogNormal as n increases



$\sigma_{\ln X} \sim COV_X$

COV_X is a good measure of variability

COV and LogNormal Distributions

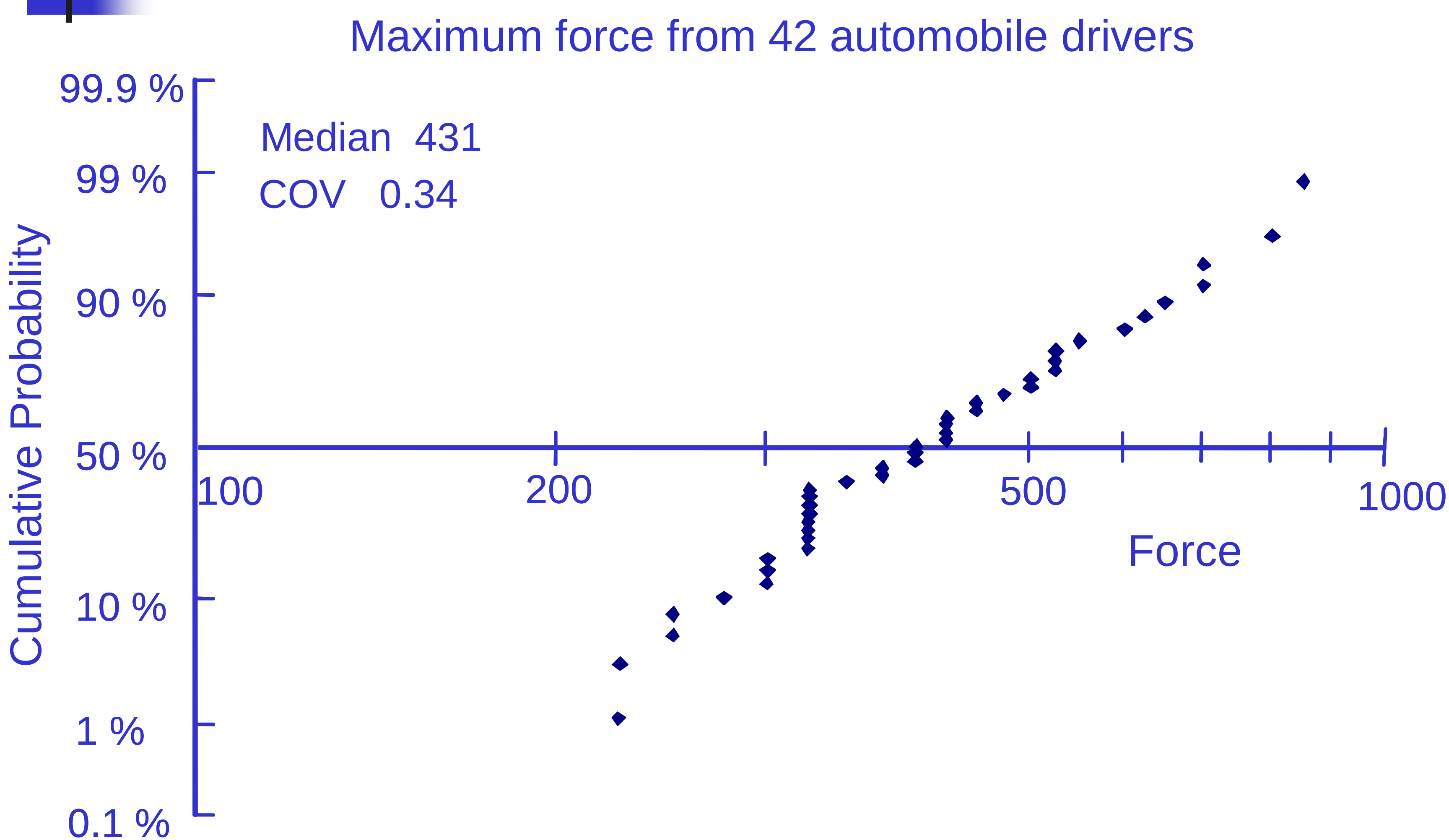
COV_X	Standard Deviation, lnx		
	1 68.3%	2 95.4%	3 99.7%
0.05	1.05	1.11	1.16
0.1	1.10	1.23	1.33
0.25	1.28	1.66	2.04
0.5	1.60	2.64	3.92
1	2.30	5.53	11.1

99.7% of the data is within a factor of ± 1.33 of the mean for a COV = 0.1

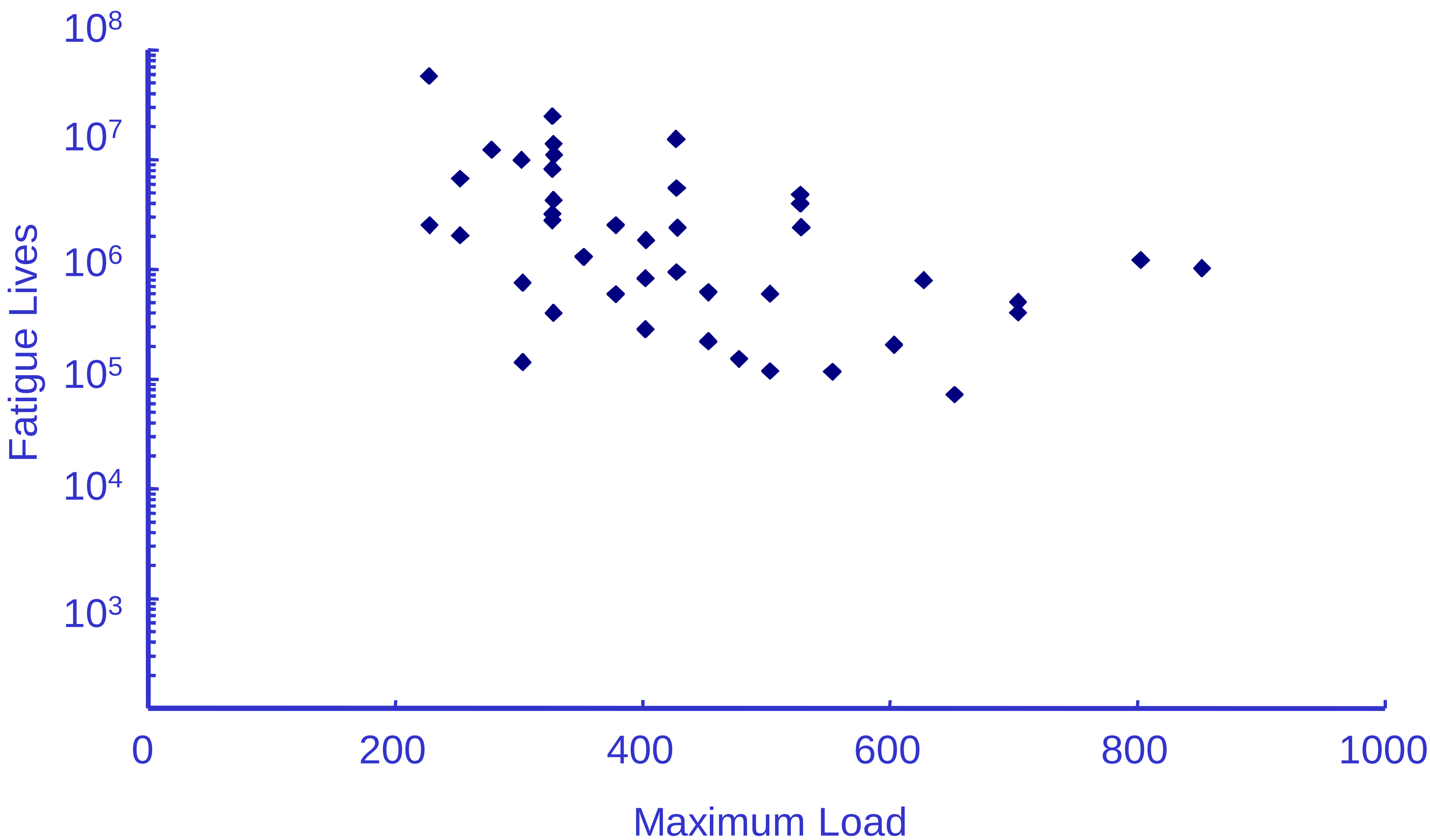
Variability in Service Loading

- Quantifying Loading Variability
  - Maximum Load
  - Load Range
  - Equivalent Stress

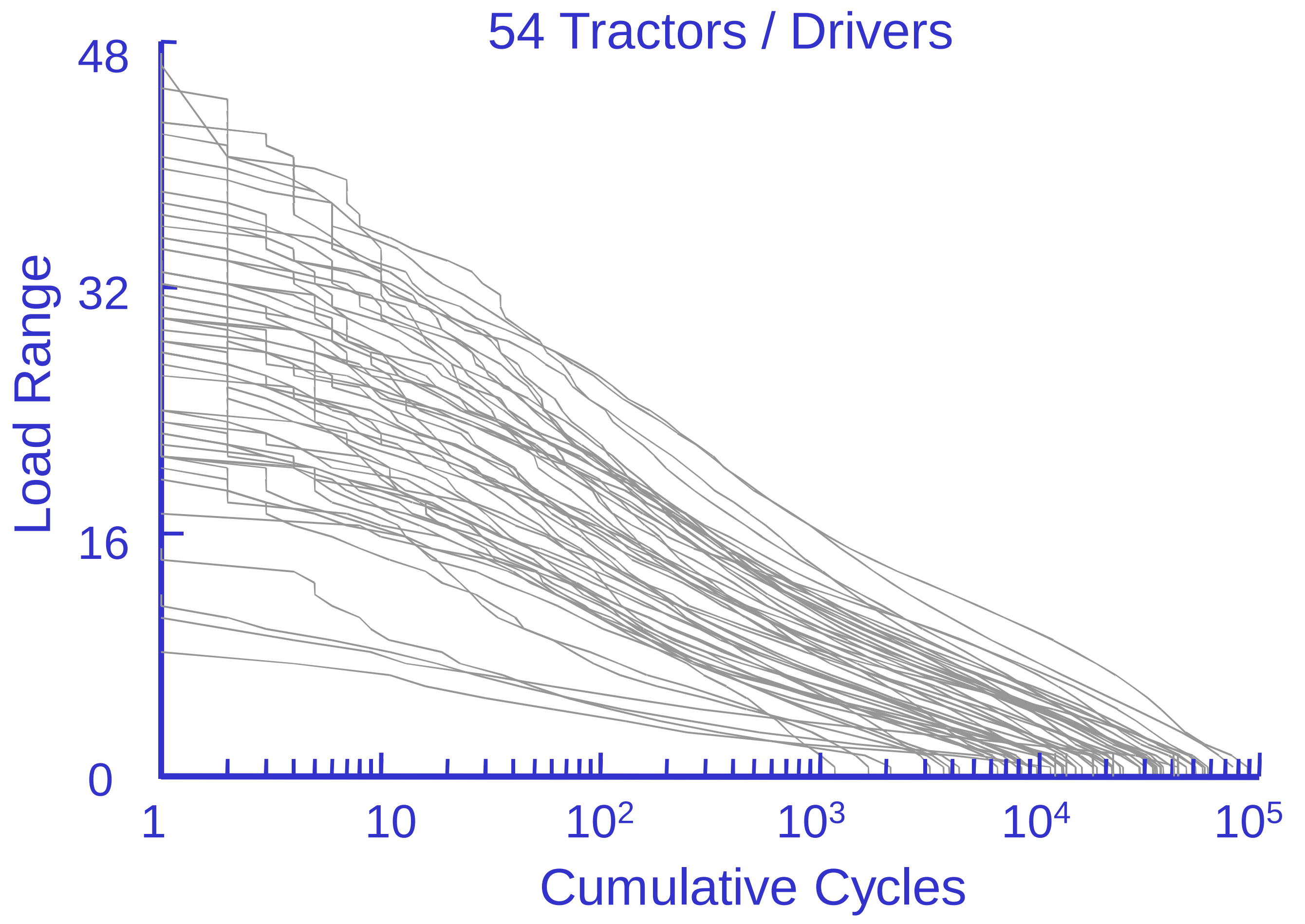
Maximum Force



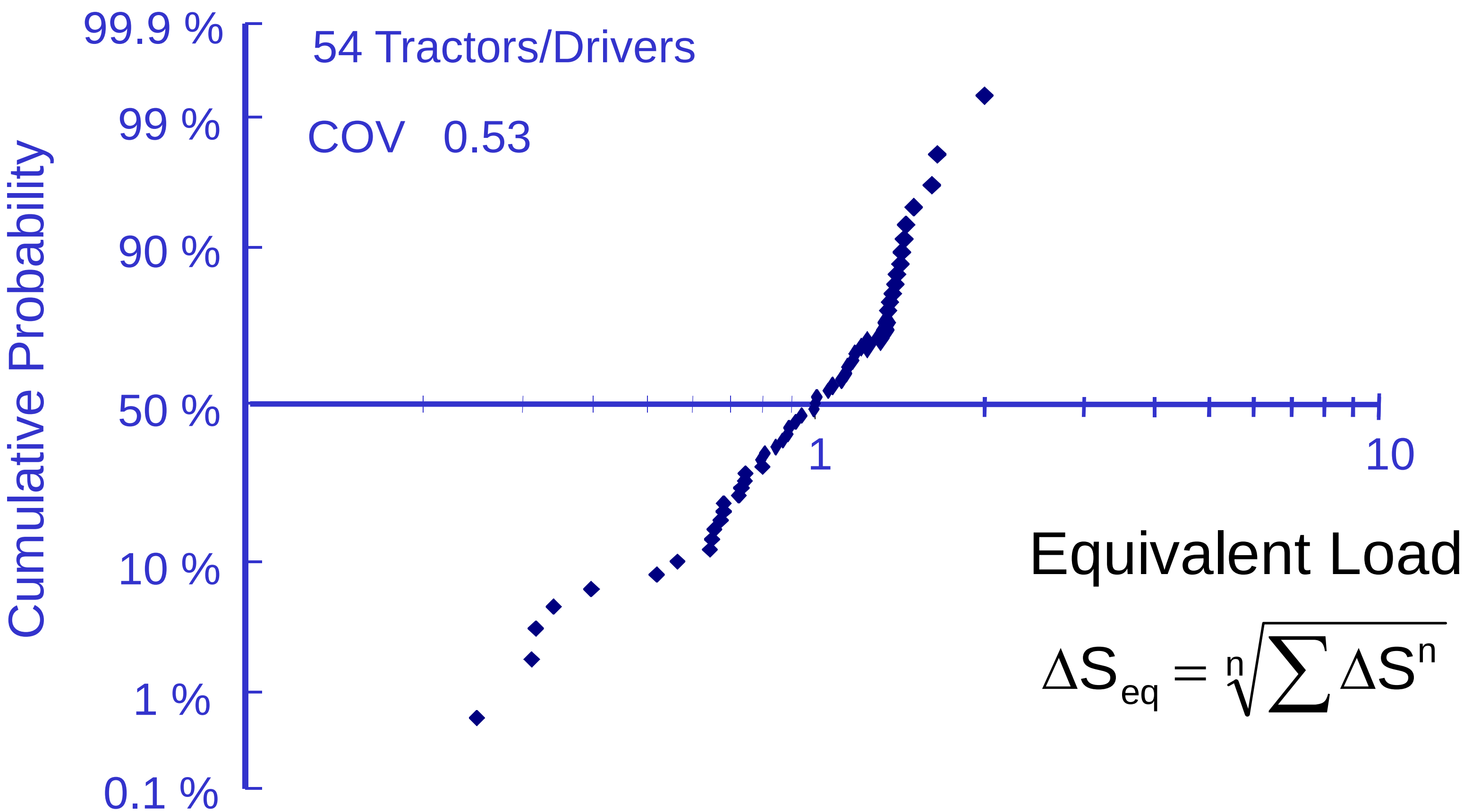
Maximum Load Correlation



Loading Variability

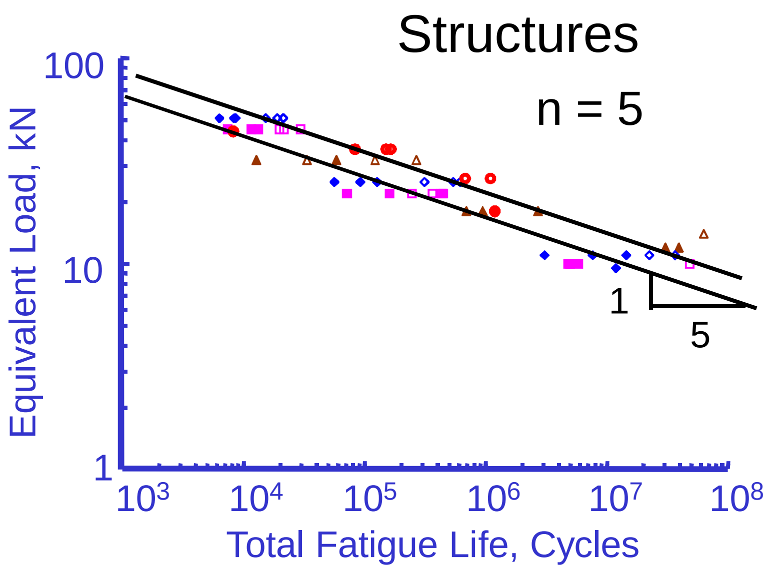


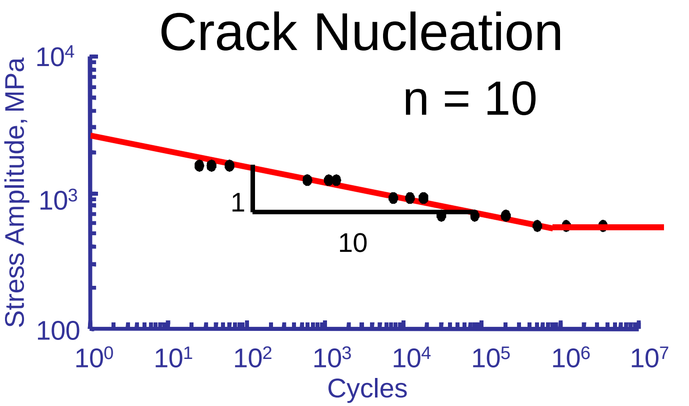
Variability in Loading

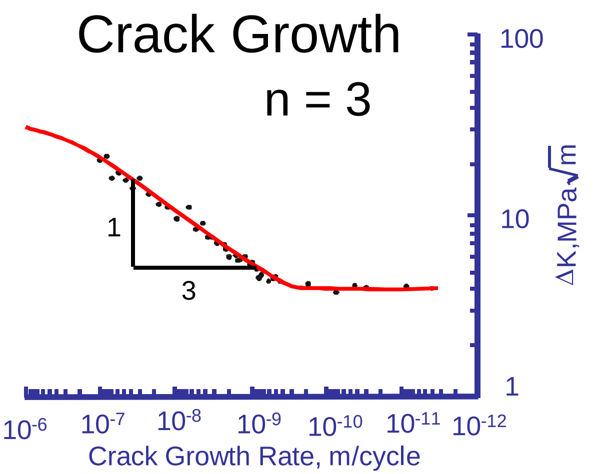


$$\Delta S_{eq} = \sqrt[n]{\sum \Delta S^n}$$

Mechanisms and Slopes

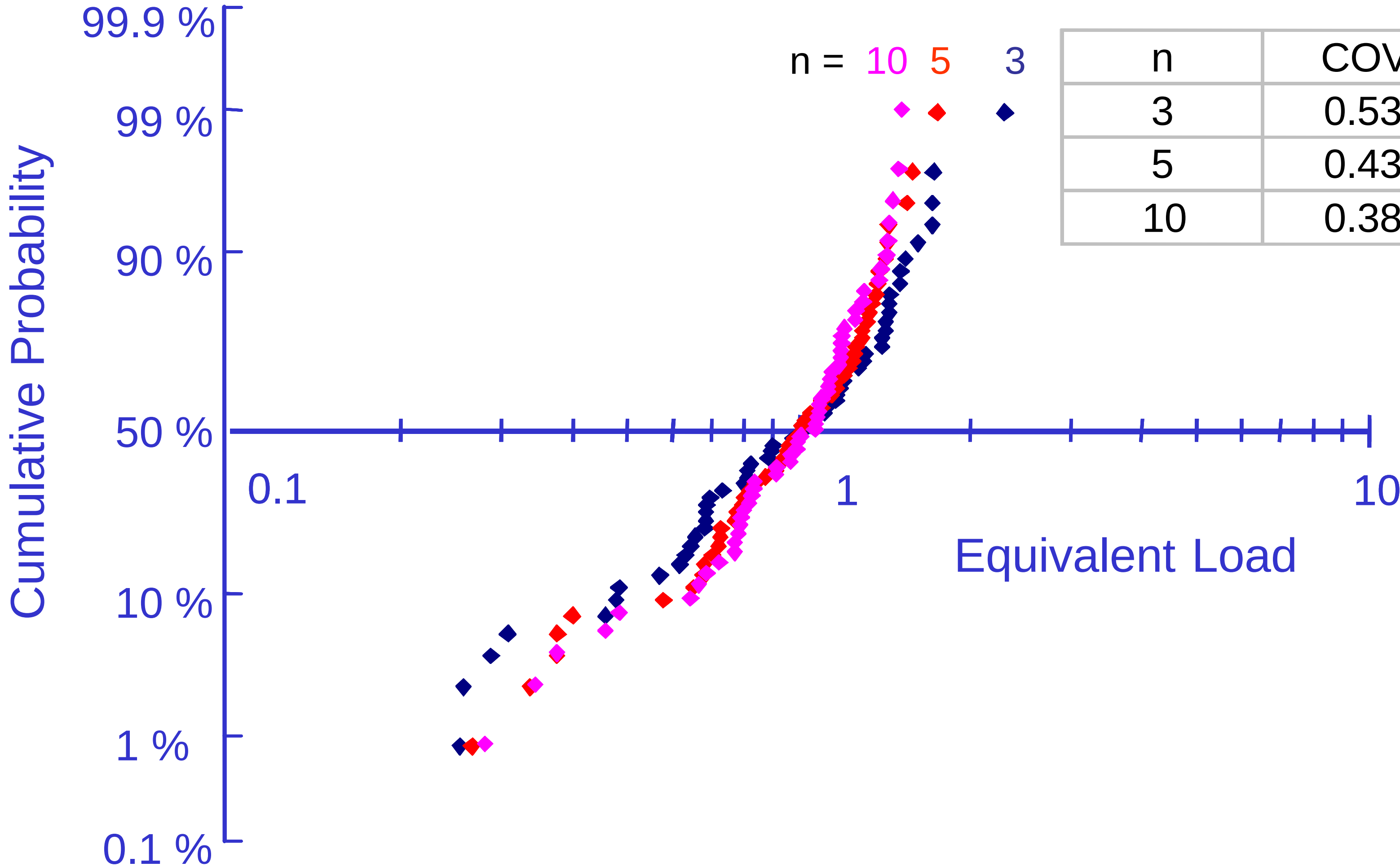






A combination of nucleation and growth

Effect of Slope on Variability

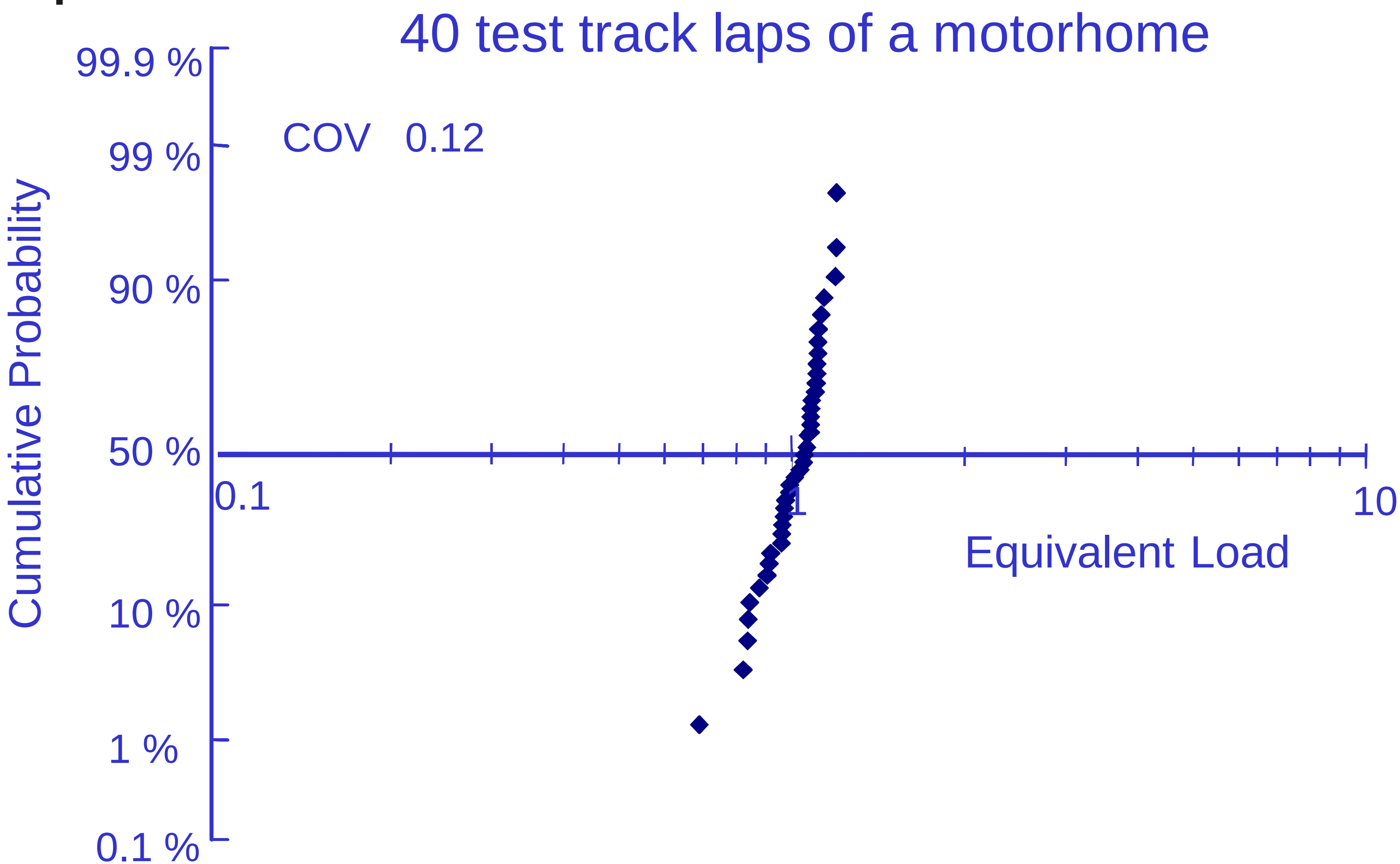


n	COV
3	0.53
5	0.43
10	0.38

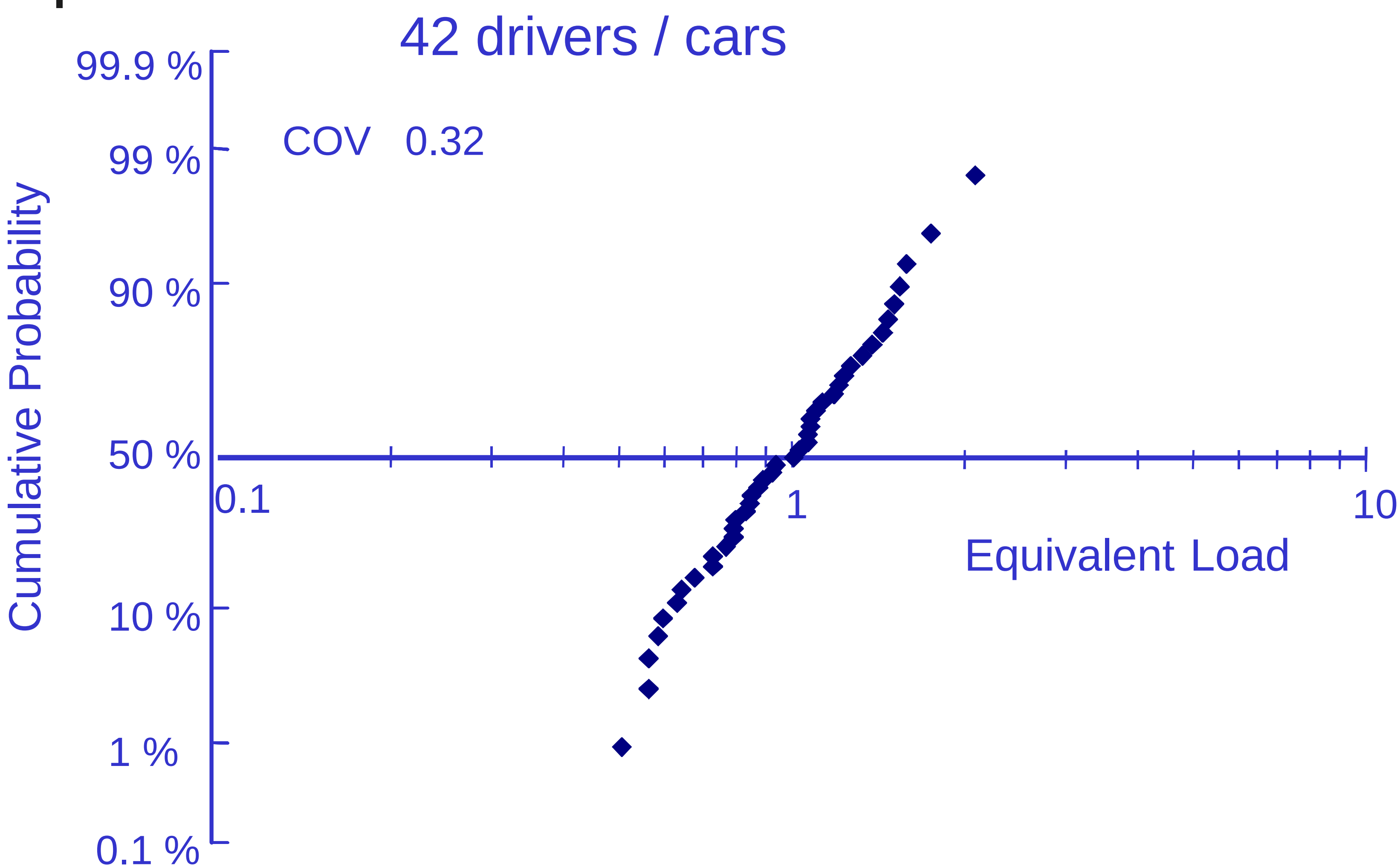
Loading History Variability

- Test Track
- Customer Service

Test Track Variability



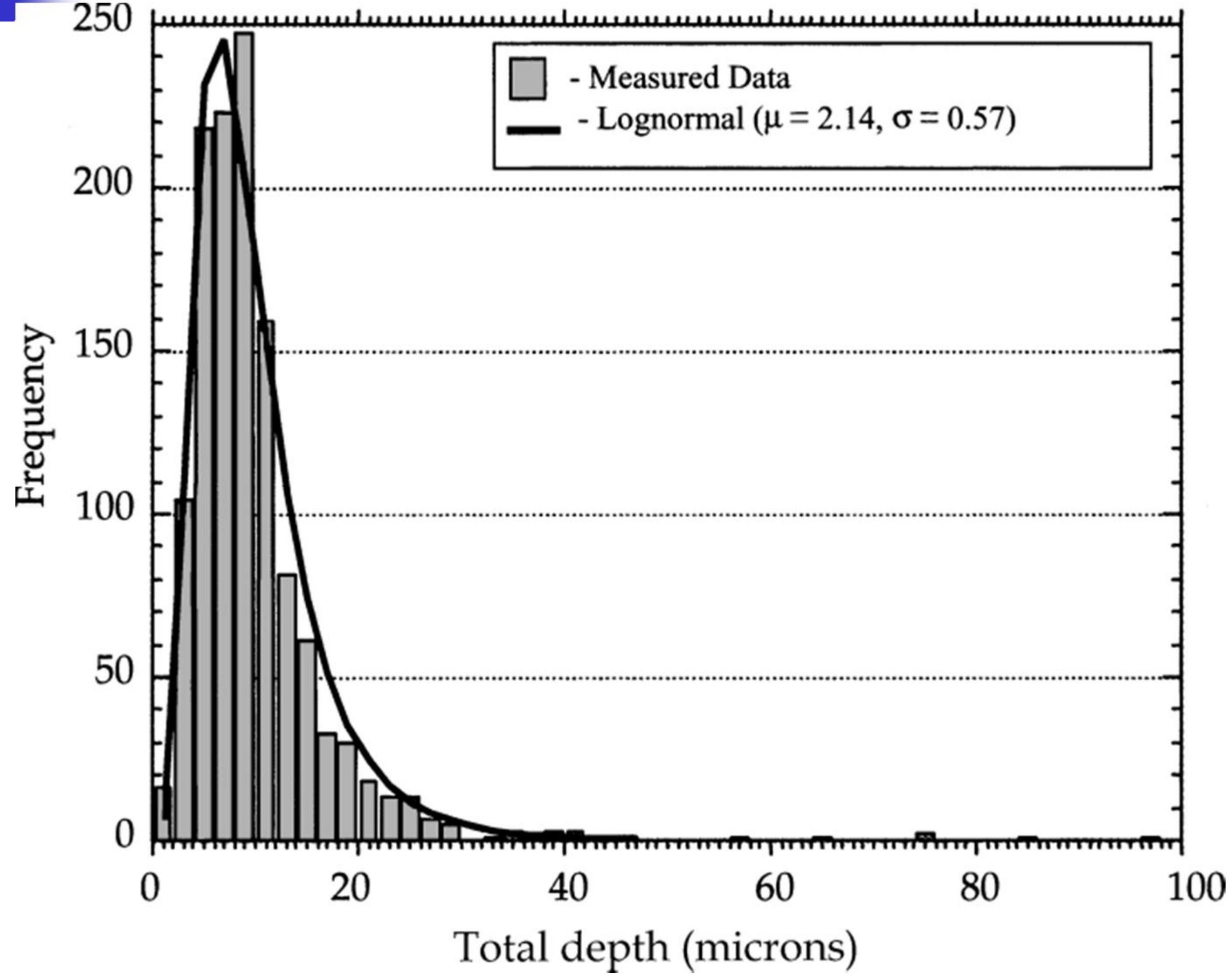
Customer Usage Variability



Variability in Environment

- Inclusions
- Pit depth

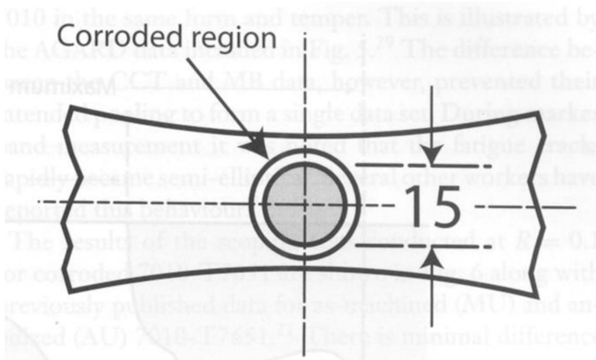
Inclusions That Initiated Cracks

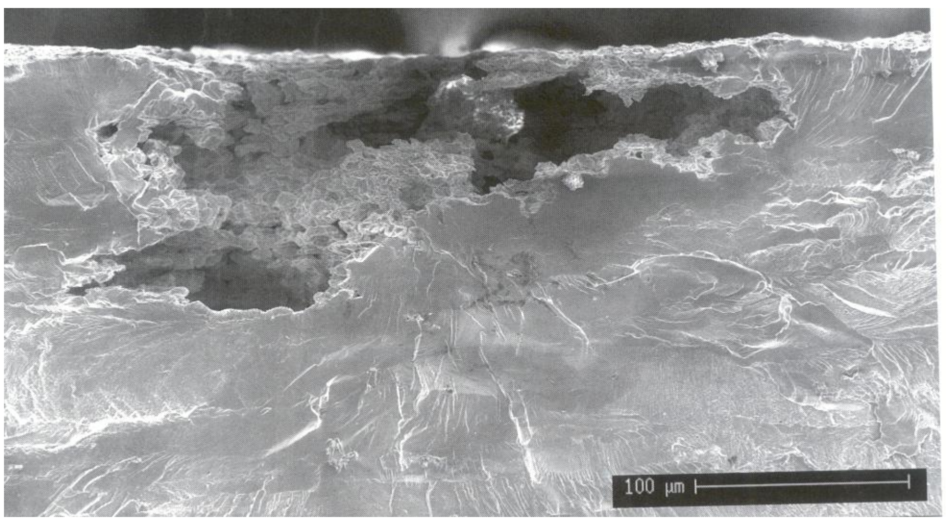


COV = 0.27

Barter, S. A., Sharp, P. K., Holden, G. & Clark, G. "Initiation and early growth of fatigue cracks in an aerospace aluminium alloy", *Fatigue & Fracture of Engineering Materials & Structures* **25** (2), 111-125.

Pits That Initiated Cracks





7010-T7651

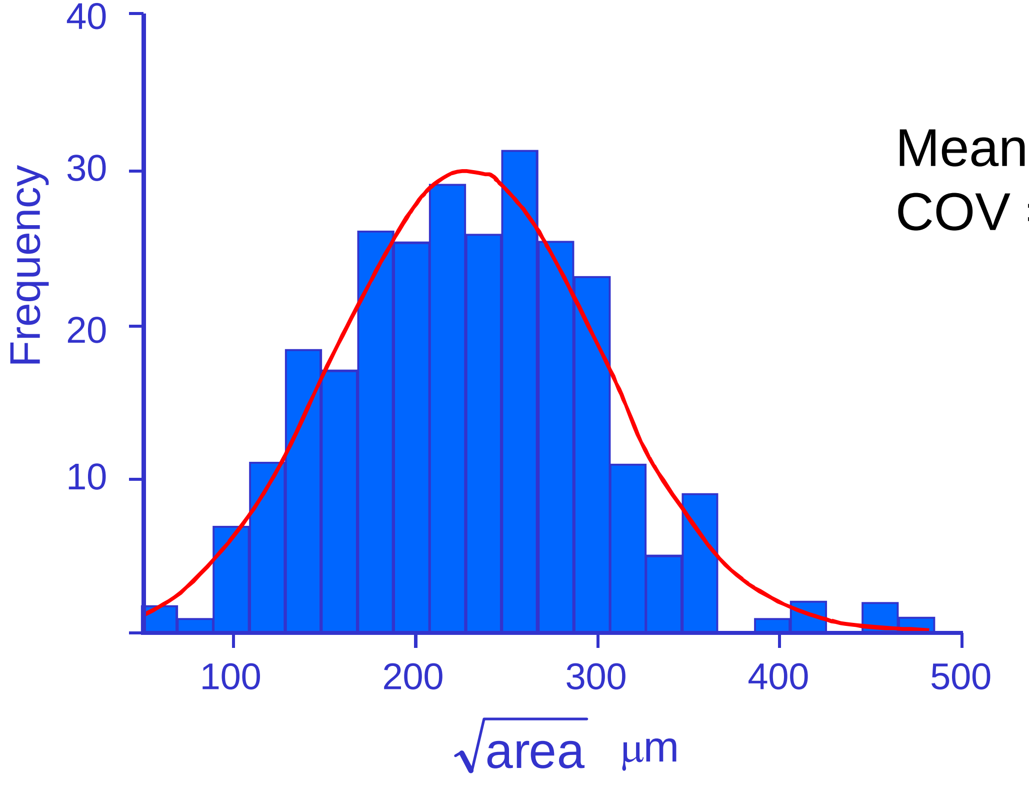
Pre-corroded specimens

300 specimens

246 failed from pits

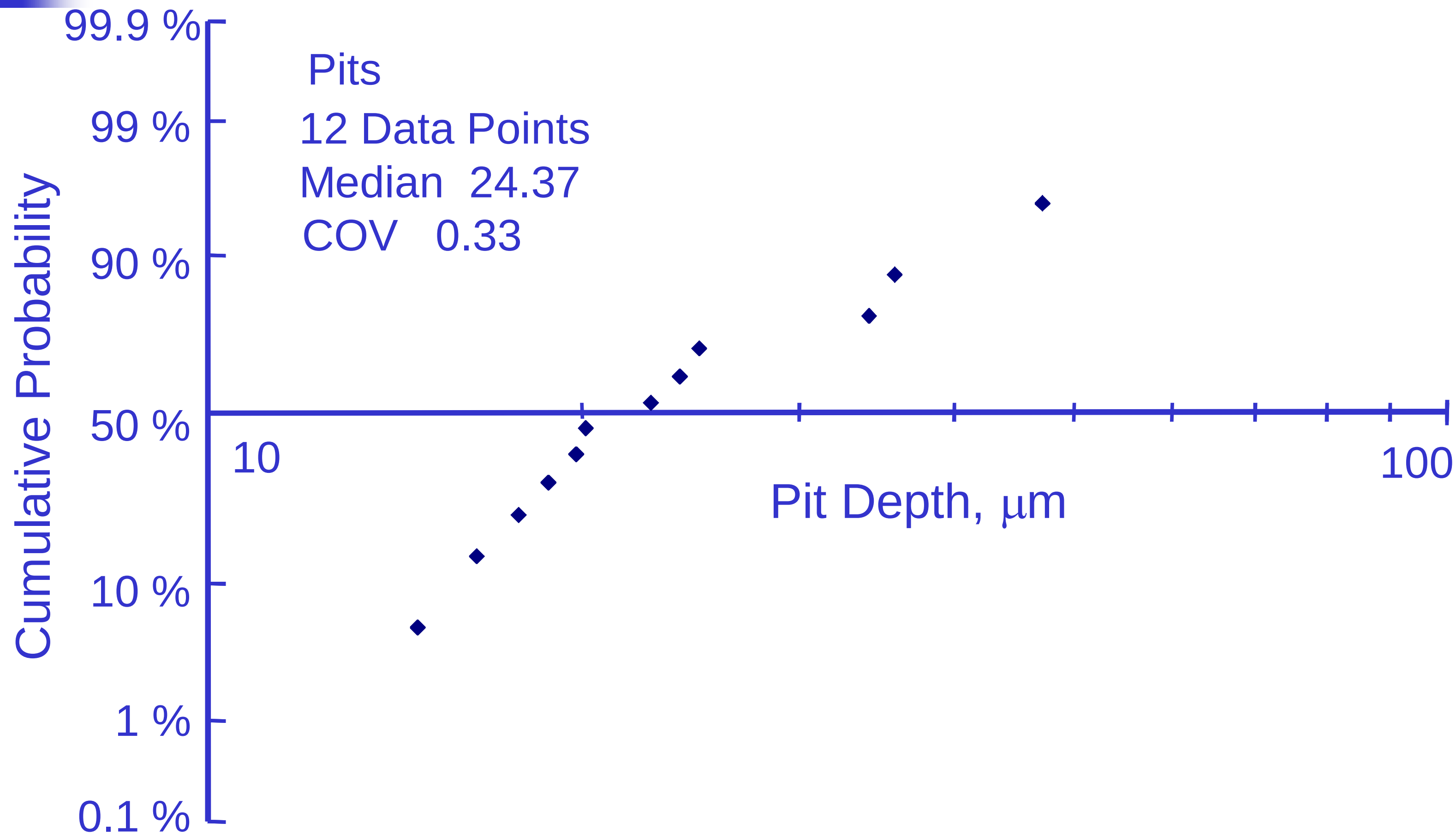
Crawford et.al."The EIFS Distribution for Anodized and Pre-corroded 7010-T7651 under Constant Amplitude Loading" *Fatigue and Fracture of Engineering Materials and Structures*, Vol. 28, No. 9 2005, 795-808

Pit Size Distribution



Mean = 230
COV = 0.32

Pit Depth Variability

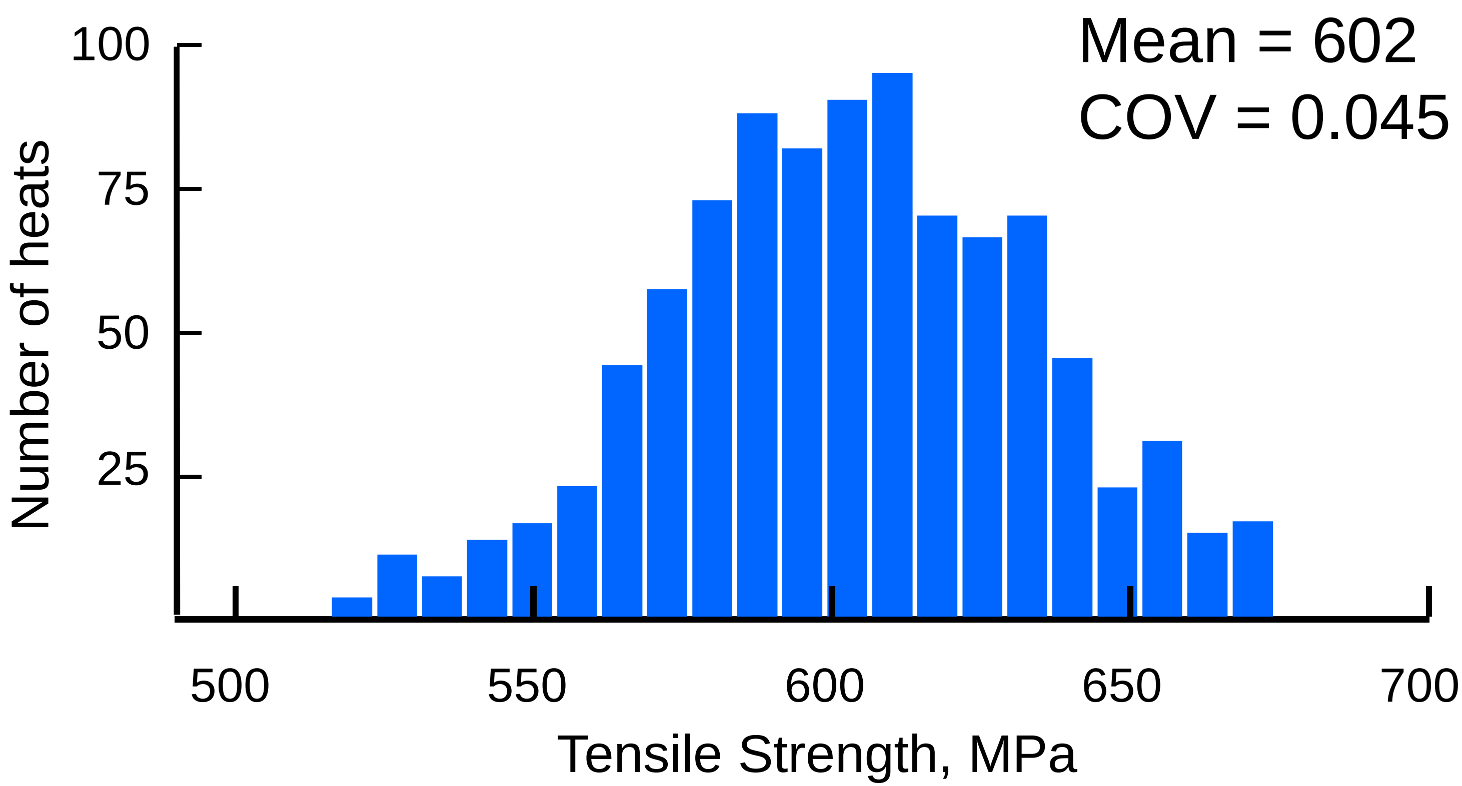


Dolly, Lee, Wei, “The Effect of Pitting Corrosion on Fatigue Life”
Fatigue and Fracture of Engineering Materials and Structures, Vol. 23, 2000, 555-560

Variability in Materials

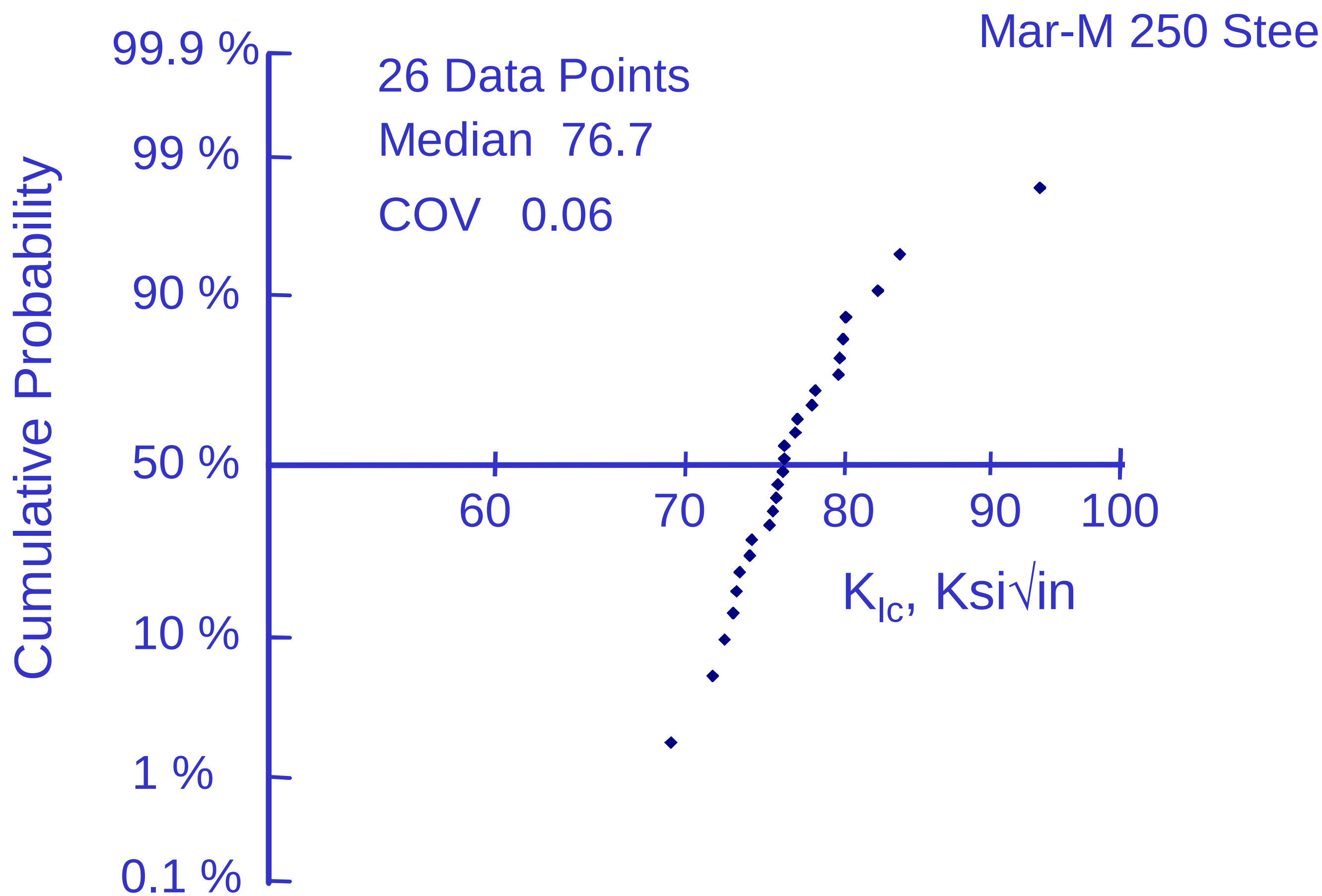
- Tensile Strength
- Fracture Toughness
- Fatigue
  - Fatigue Strength
  - Fatigue Life
- Strain-Life
- Crack Growth

Tensile Strength - 1035 Steel



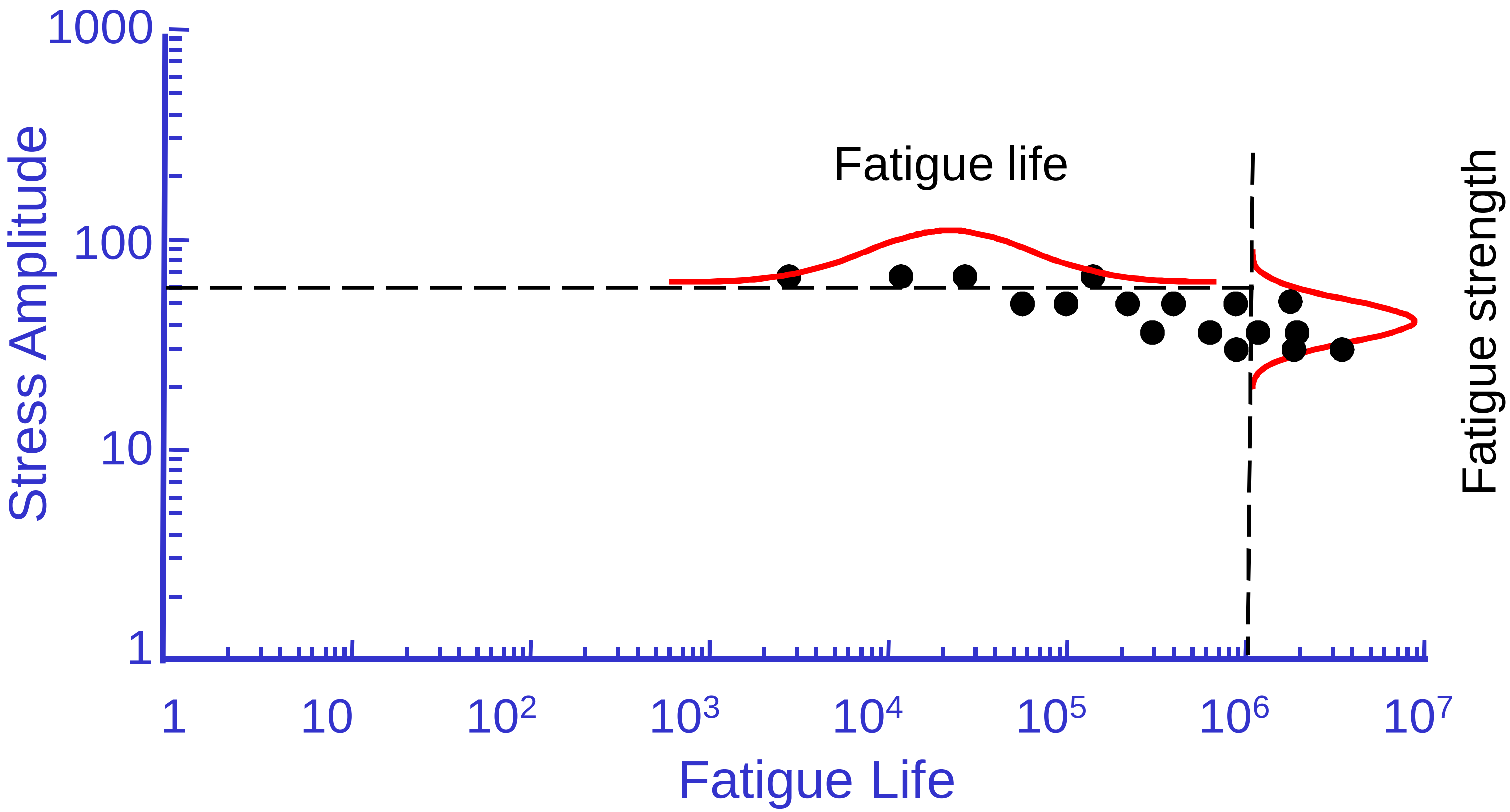
Metals Handbook, 8th Edition, Vol. 1, p64

Fracture Toughness

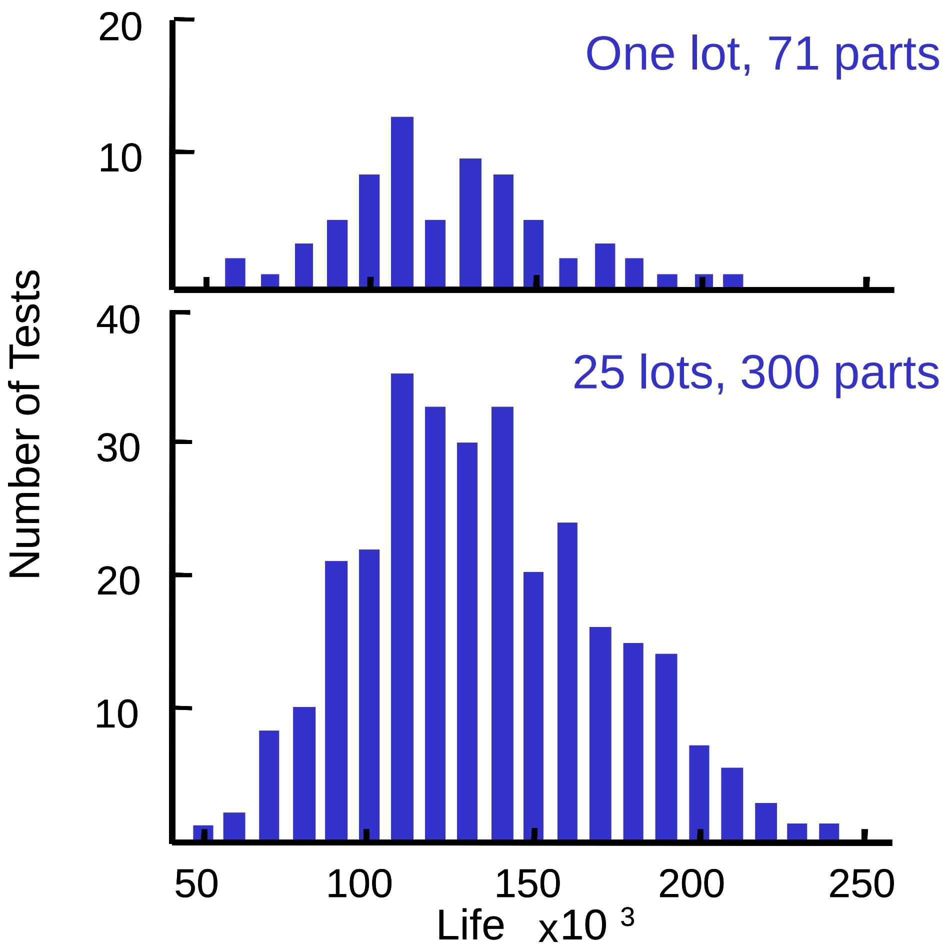


Kies, J.A., Smith, H.L., Romine, H.E. and Bernstein, H, “Fracture Testing of Weldments”, ASTM STP 381, 1965, 328-356

Fatigue Variability



Fatigue Life Variability



Production torsion bars

5160H steel

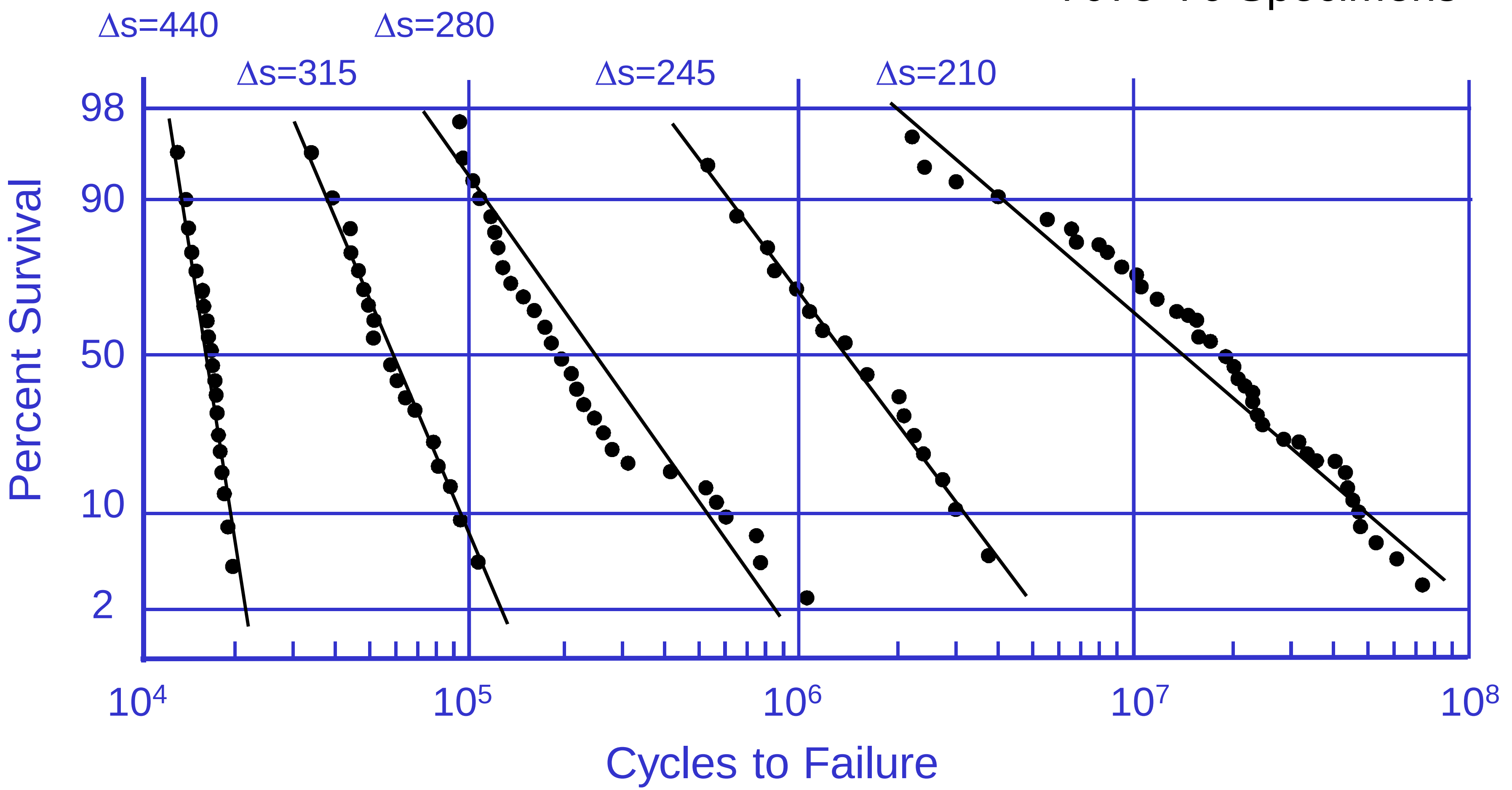
μ_x = 123,000 cycles
COV = 0.25

μ_x = 134,000 cycles
COV = 0.27

Metals Handbook, 8th Edition, Vol. 1, p219

Statistical Variability of Fatigue Life

7075-T6 Specimens



Sinclair and Dolan, "Effect of Stress Amplitude on the Variability in Fatigue Life of 7075-T6 Aluminum Alloy" Transactions ASME, 1953

COV vs Fatigue Life

ΔS	$\overline{X}$	COV
440	14,000	0.12
315	25,000	0.38
280	220,000	0.70
245	1,200,000	0.67
210	12,000,000	1.39

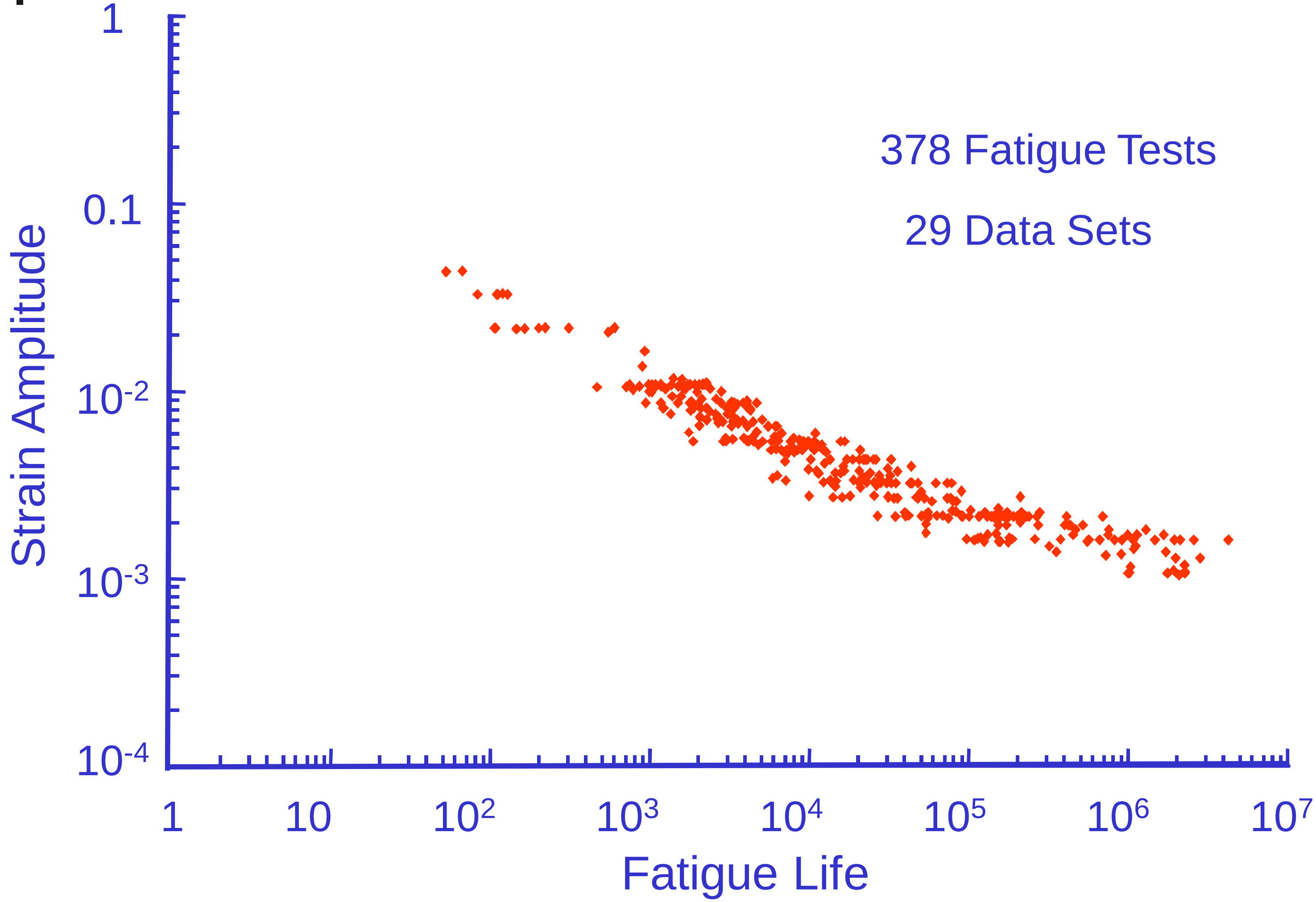
Variability in Fatigue Strength

$$\frac{\Delta S}{2} = S_f^{'}(N_f)^b \quad b \approx -0.085$$

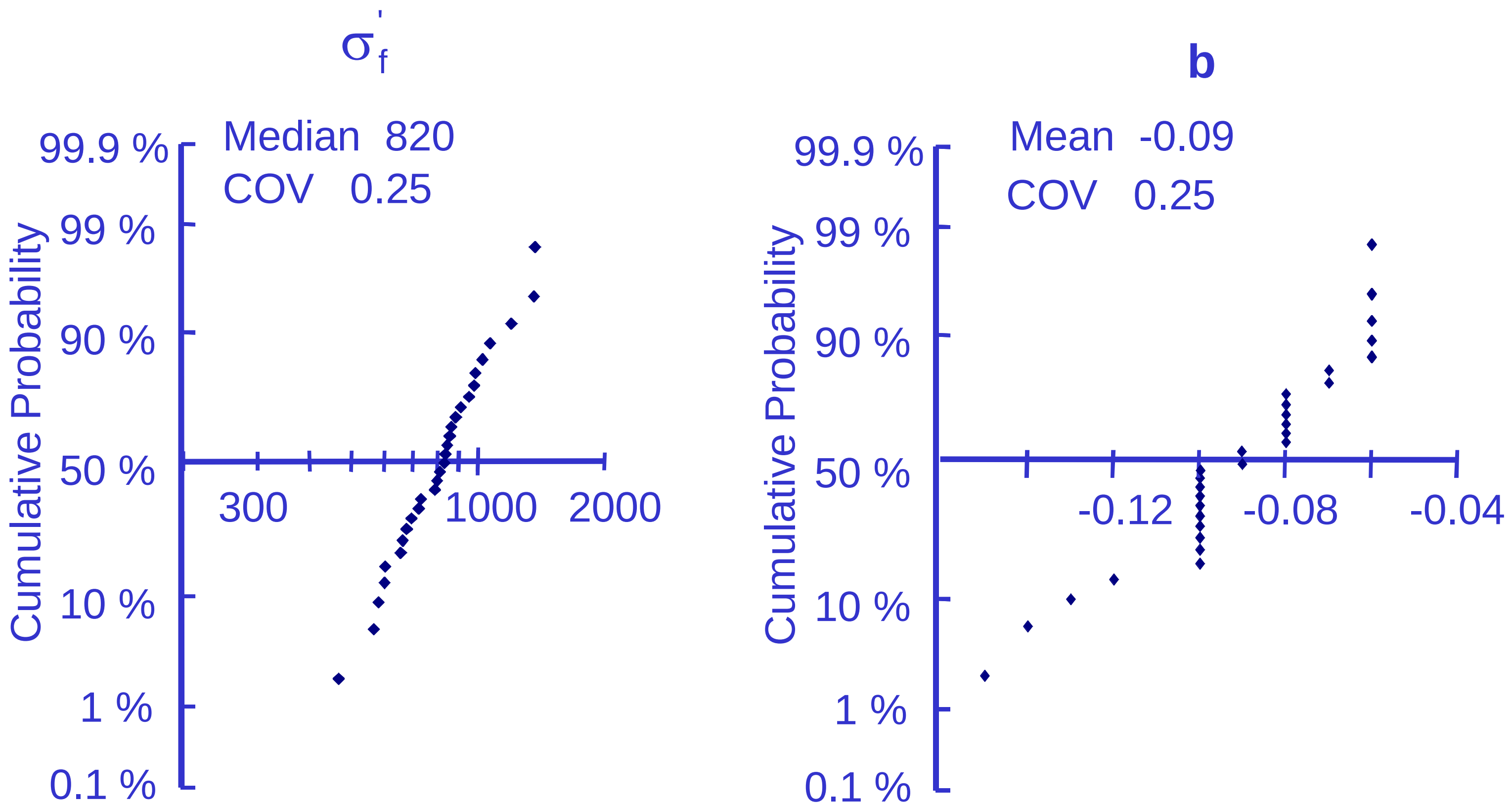
$$\text{COV} \quad C = \sqrt{\prod_{i=1}^{n}\left(1 + C_{X_i}^{\;2}\right)^{a_i^2} - 1}$$

$$C_{S_f^{'}} = \sqrt{\left(1 + 1.39^2\right)^{(-.085)^2} - 1} = 0.088$$

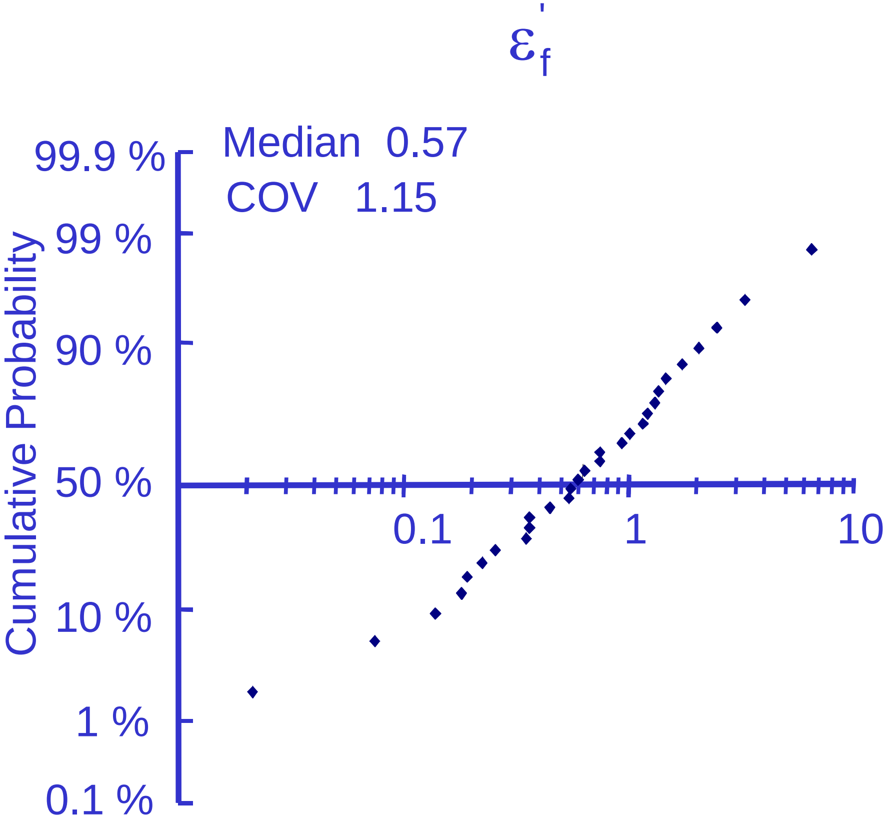
Strain Life Data for 950X Steel

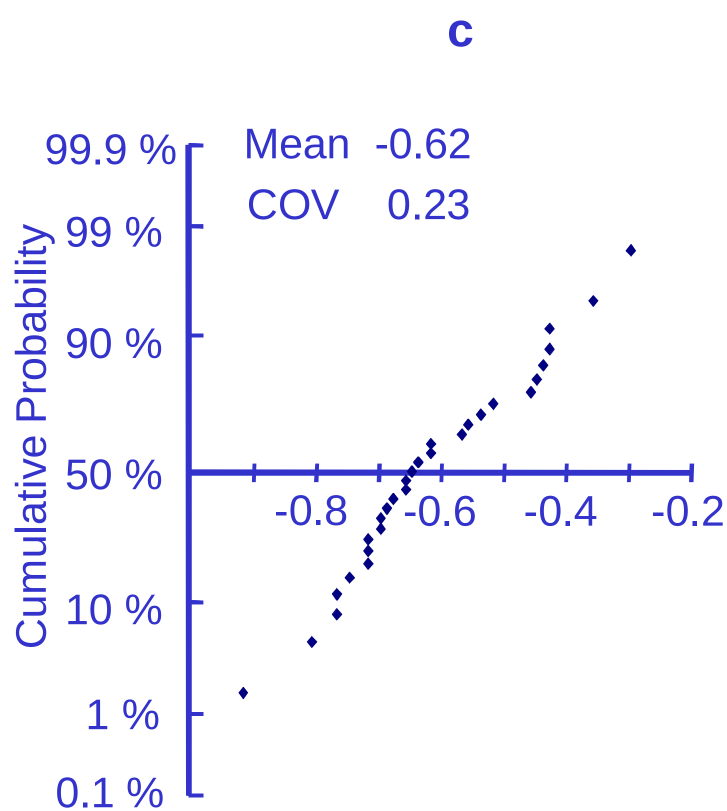


29 Individual Data Sets



29 Individual Data Sets (continued)

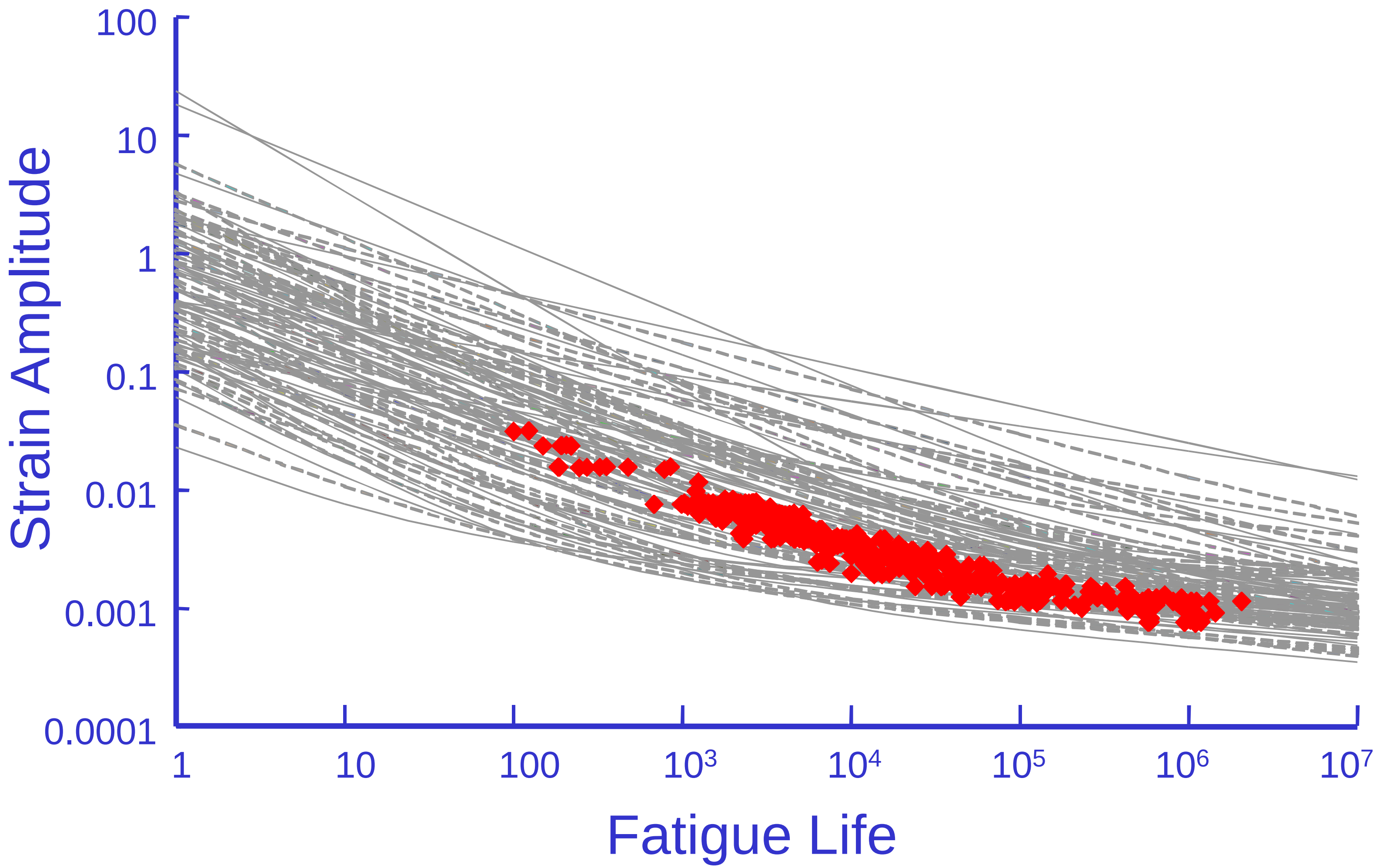




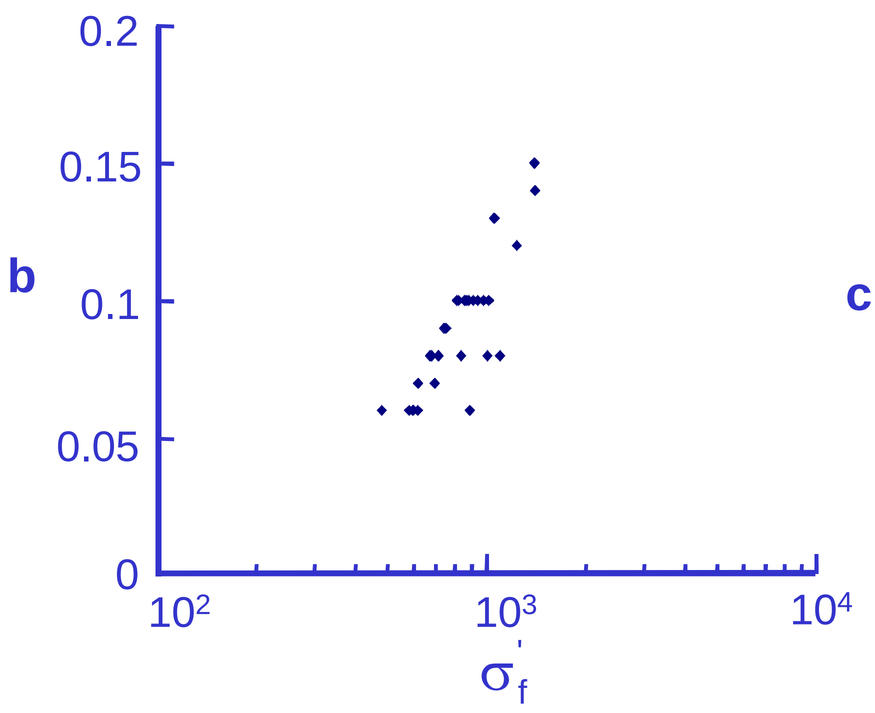
Input Data Simulation

$$\frac{\Delta\varepsilon}{2} = \frac{\sigma_f^{'}\left(L,\mu_{\sigma_f},\sigma_{\sigma_f}\right)}{E}\left(2N_f\right)^{b\left(N,\mu_b,\sigma_b\right)} + \varepsilon_f^{'}\left(L,\mu_{\varepsilon_f},\sigma_{\varepsilon_f}\right)\left(2N_f\right)^{c\left(N,\mu_b,\sigma_b\right)}$$

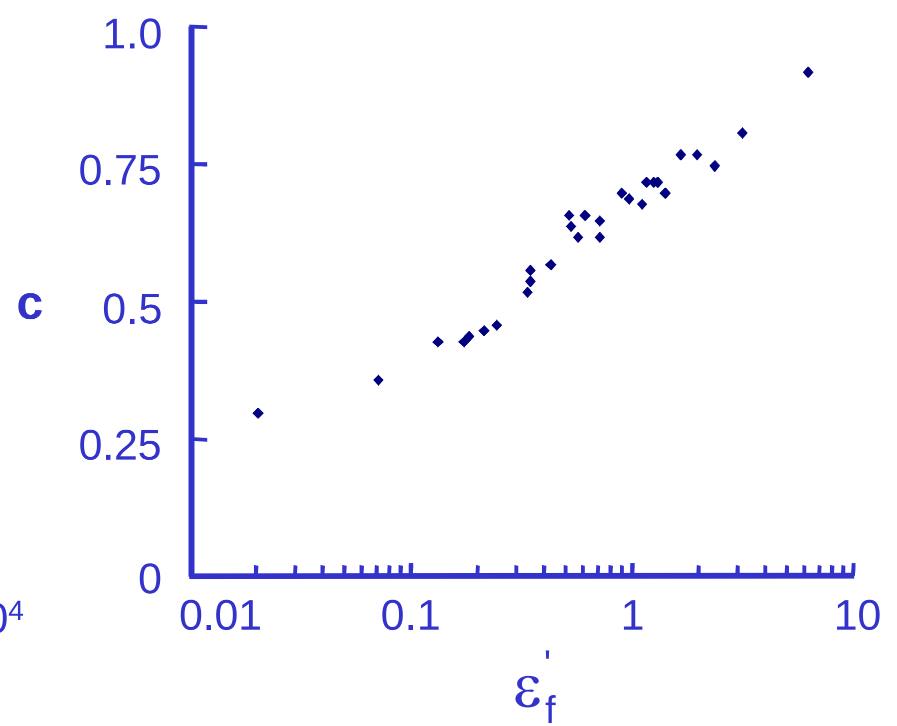
Simulation Results



Correlation



ρ = -0.828



ρ = -0.976

Generating Correlated Data

$$z_1 = \Phi(\text{ rand() })\qquad z_1 = N(0,1)$$

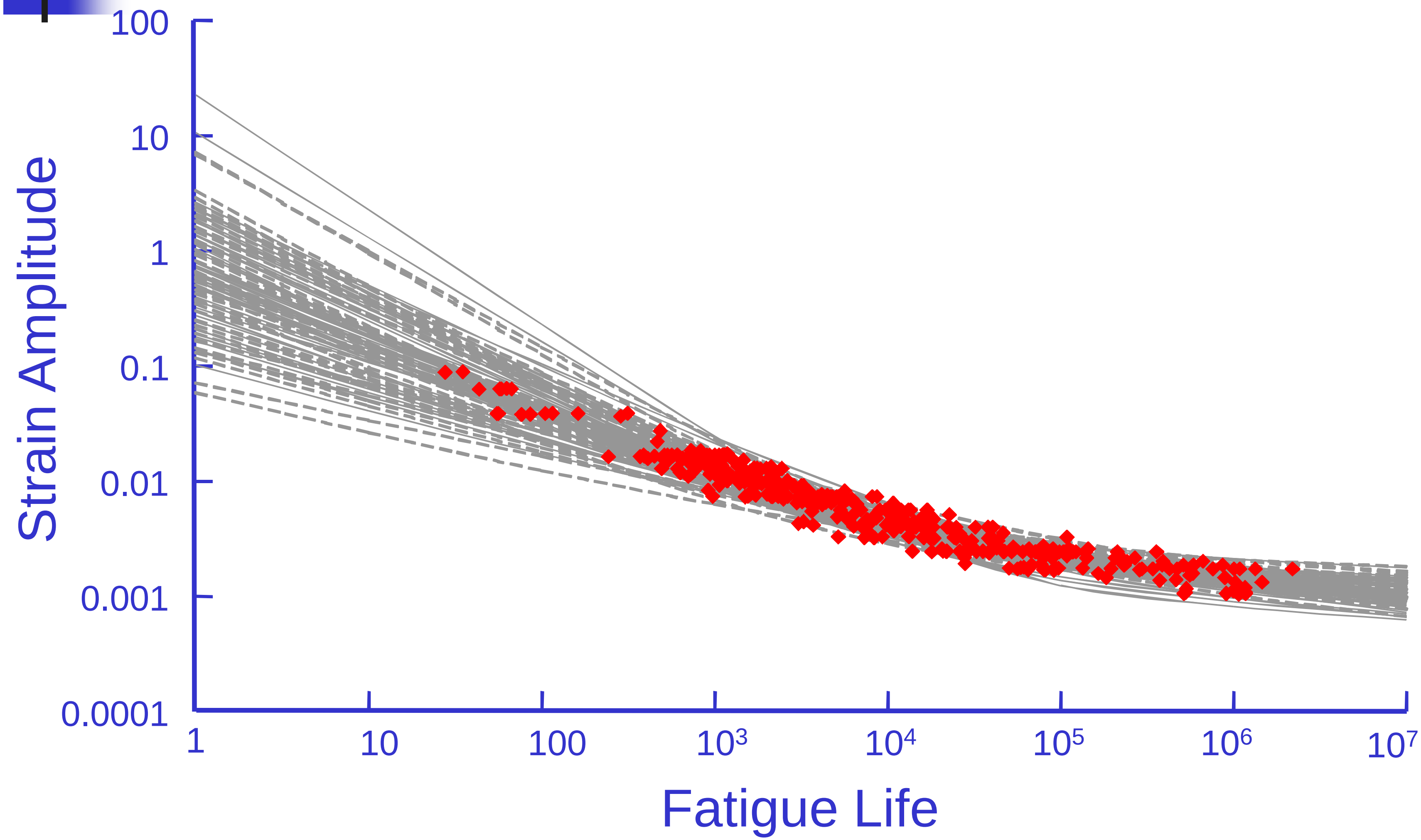
$$z_2 = \Phi(\text{ rand() })$$

$$z_3 = z_1\rho + z_2\sqrt{1-\rho^2}$$

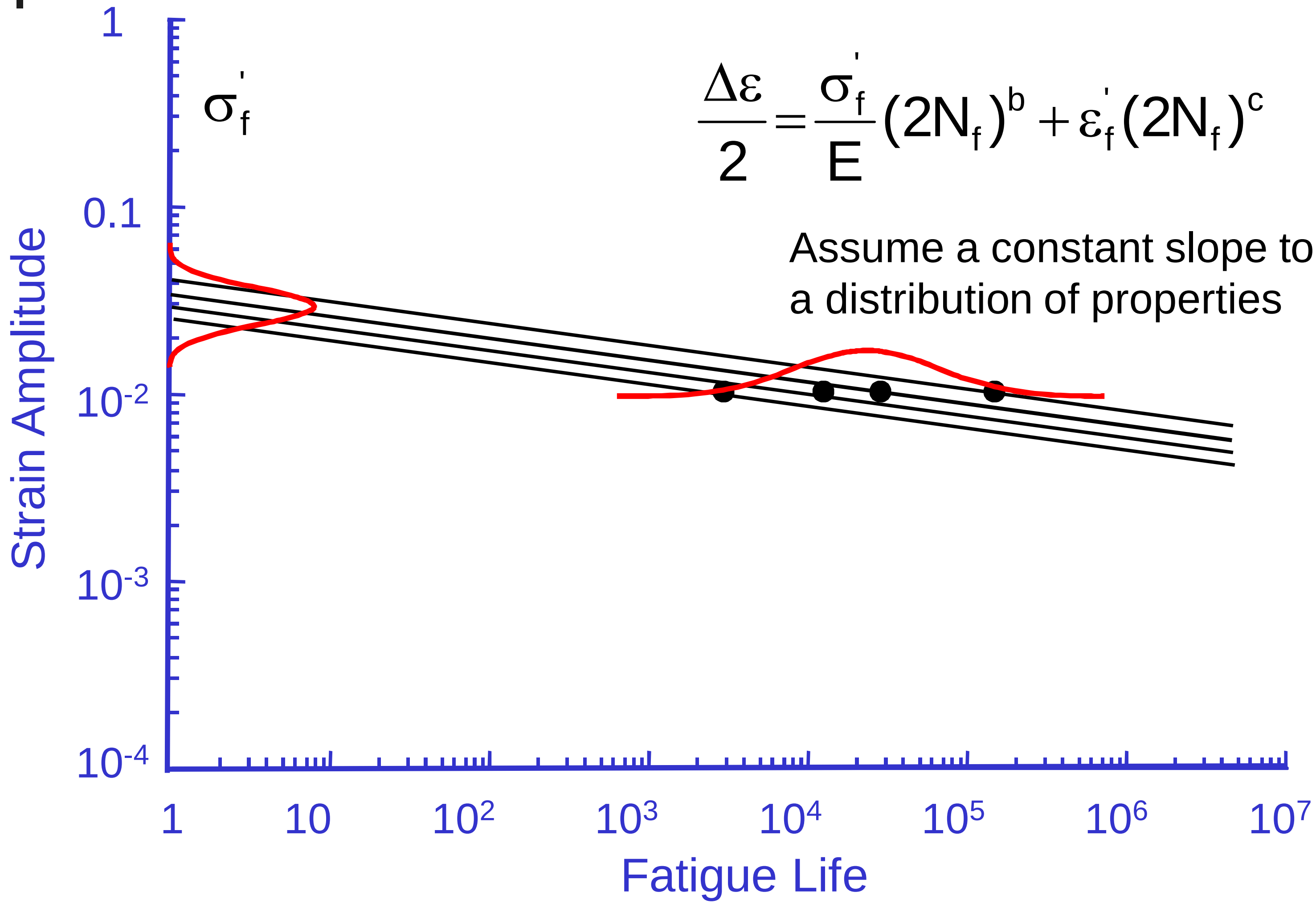
$$\sigma_f' = \exp(\mu_{\ln\sigma_f'} + \sigma_{\ln\sigma_f'} z_1)$$

$$b = \mu_b + \sigma_b z_3$$

Correlated Properties

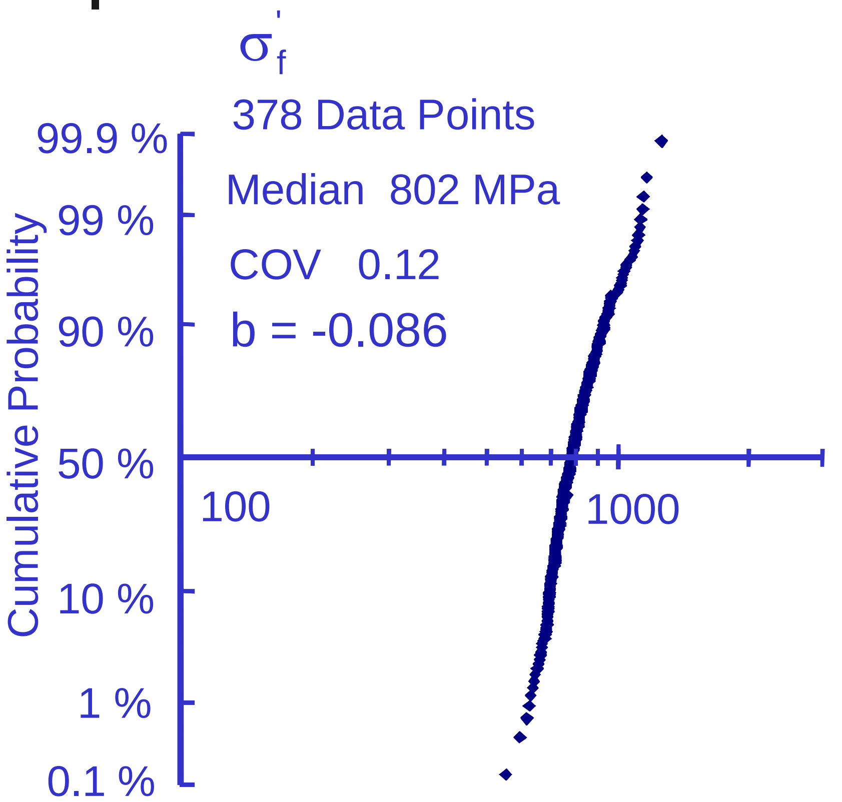


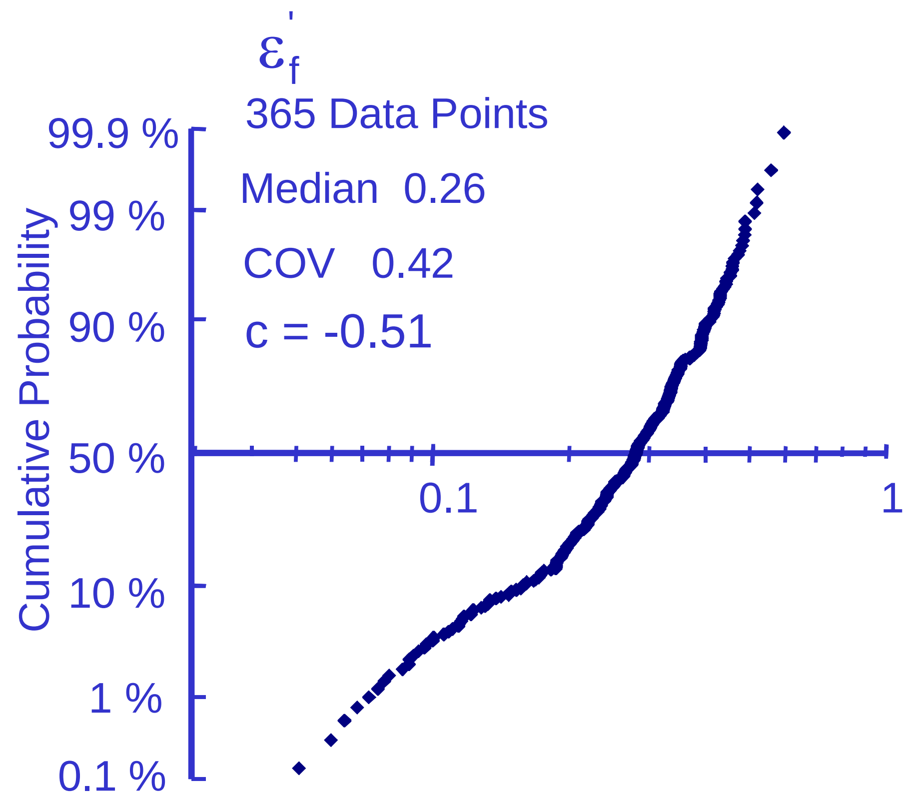
Curve Fitting



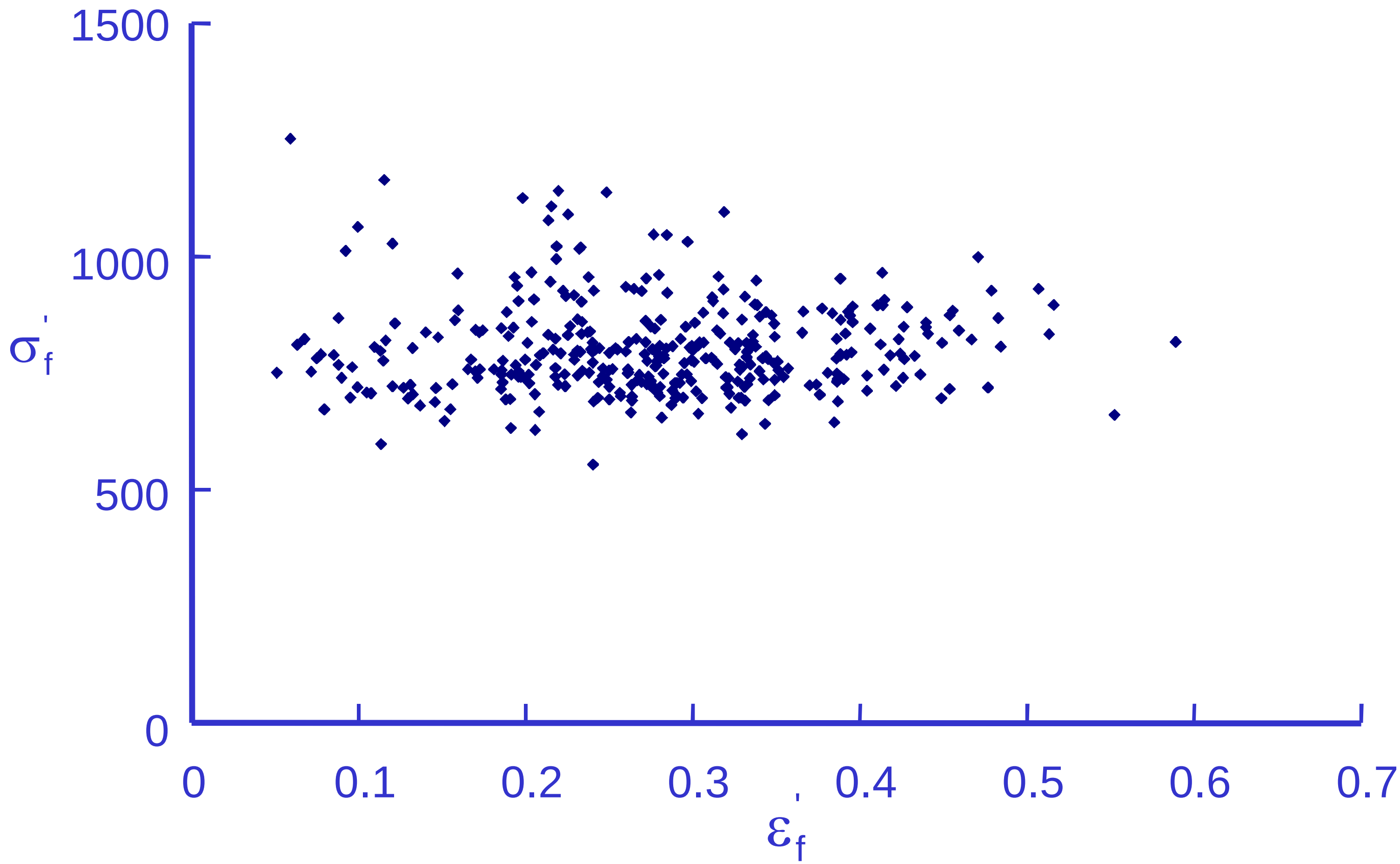
$$\frac{\Delta\varepsilon}{2} = \frac{\sigma_f'}{E}(2N_f)^b + \varepsilon_f'(2N_f)^c$$

Assume a constant slope to get a distribution of properties

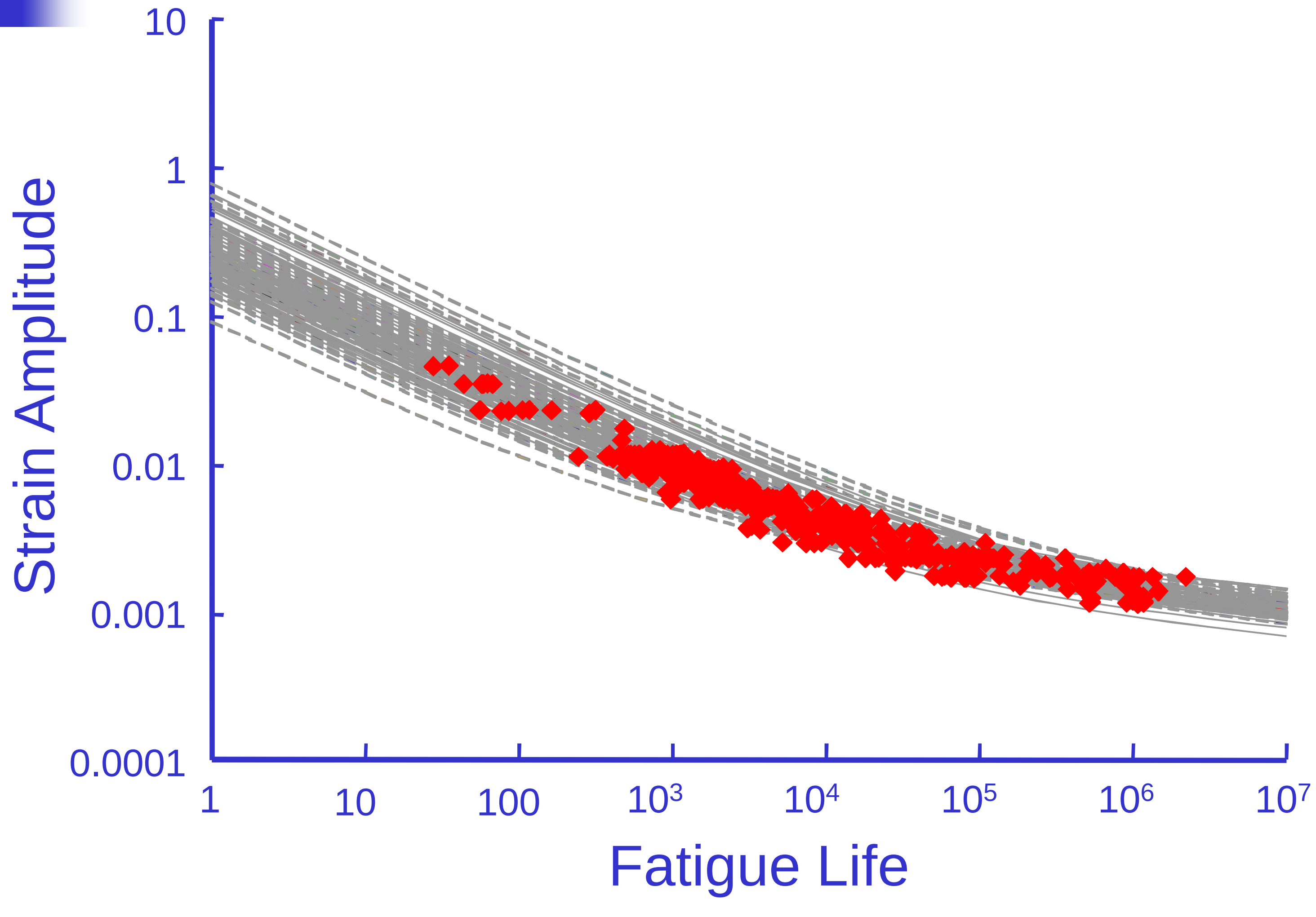

σ'f
378 Data Points
Median 802 MPa
COV 0.12
b = -0.086
Cumulative Probability
99.9 %
99 %
90 %
50 %
10 %
1 %
0.1 %
100
1000


ε'f
365 Data Points
Median 0.26
COV 0.42
c = -0.51
Cumulative Probability
99.9 %
99 %
90 %
50 %
10 %
1 %
0.1 %
0.1
1

Correlation

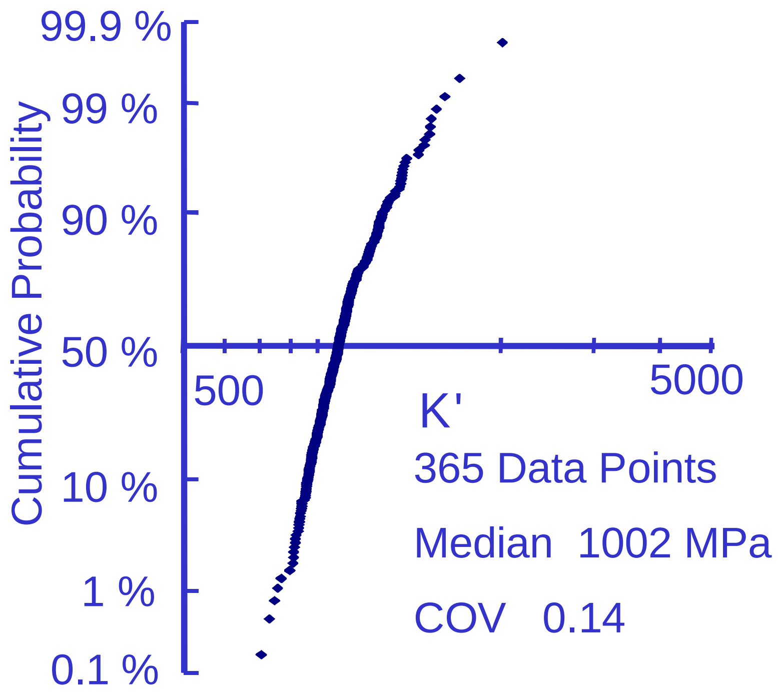


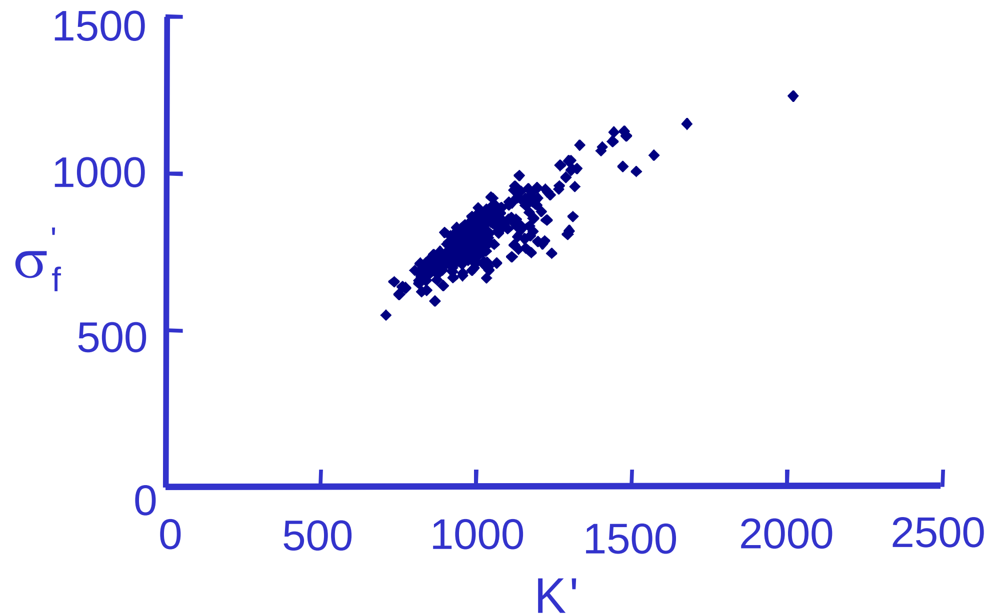
Simulation



Strain Amplitude
10
1
0.1
0.01
0.001
0.0001
1
10
100
10^3
10^4
10^5
10^6
10^7
Fatigue Life

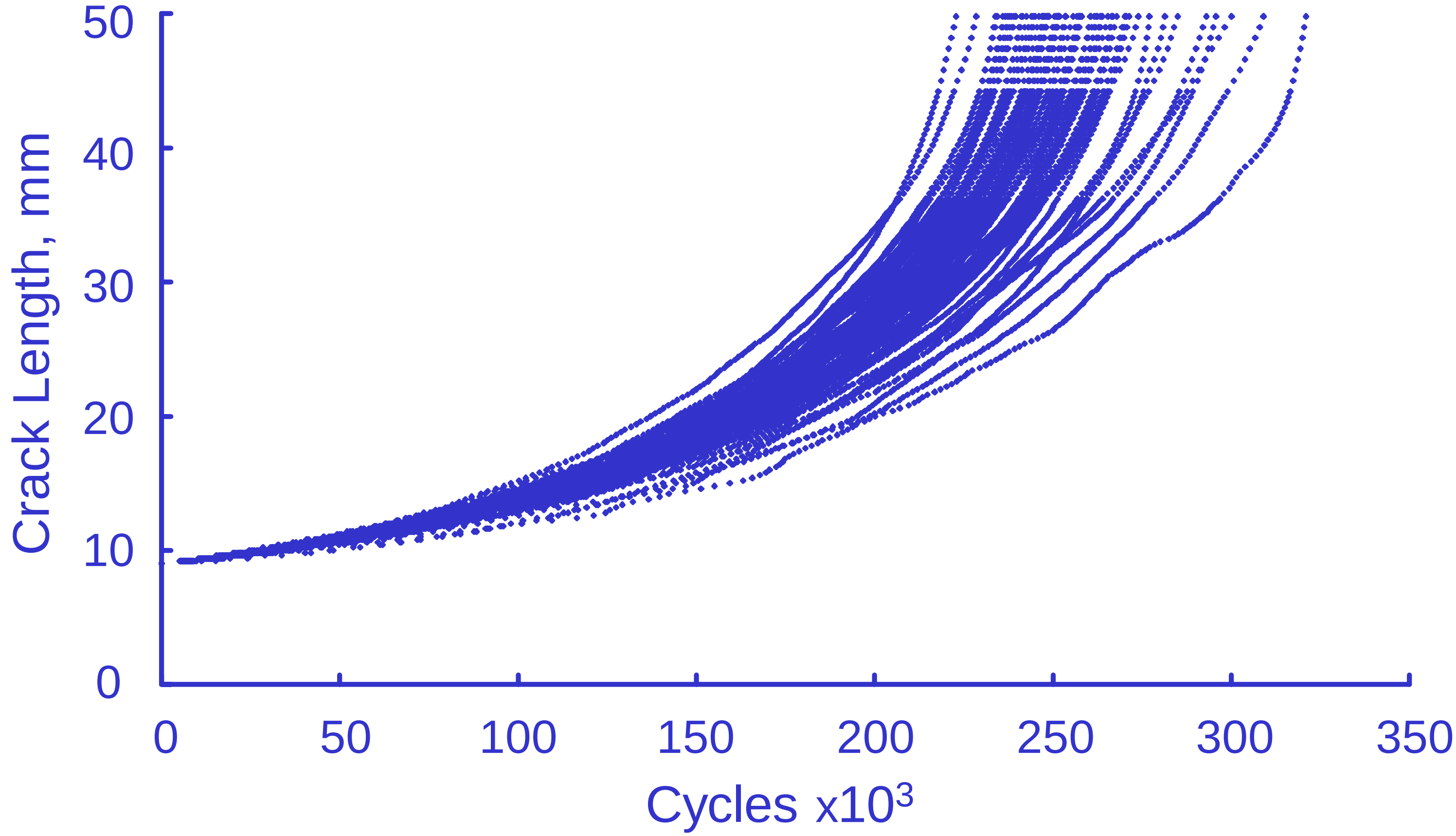
Strength Coefficient





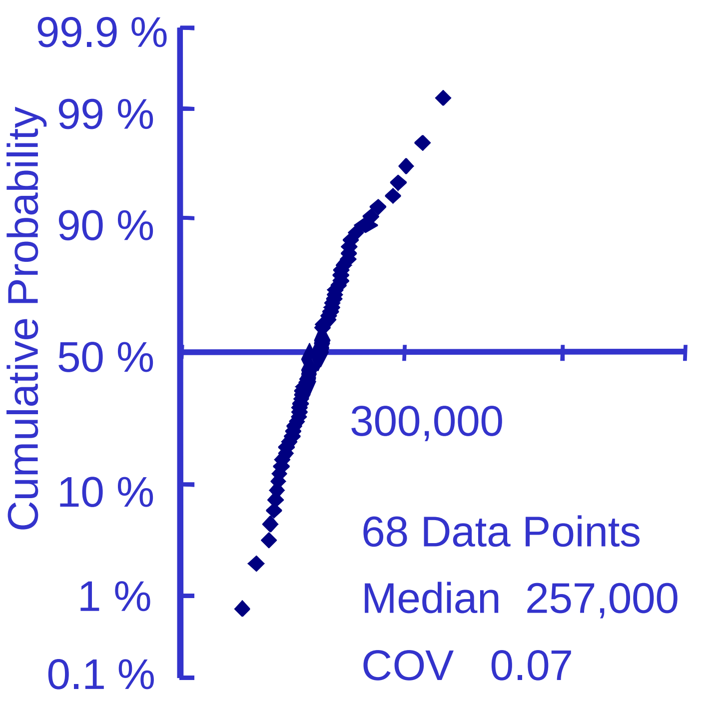
$\rho = 0.863$

Crack Growth Data

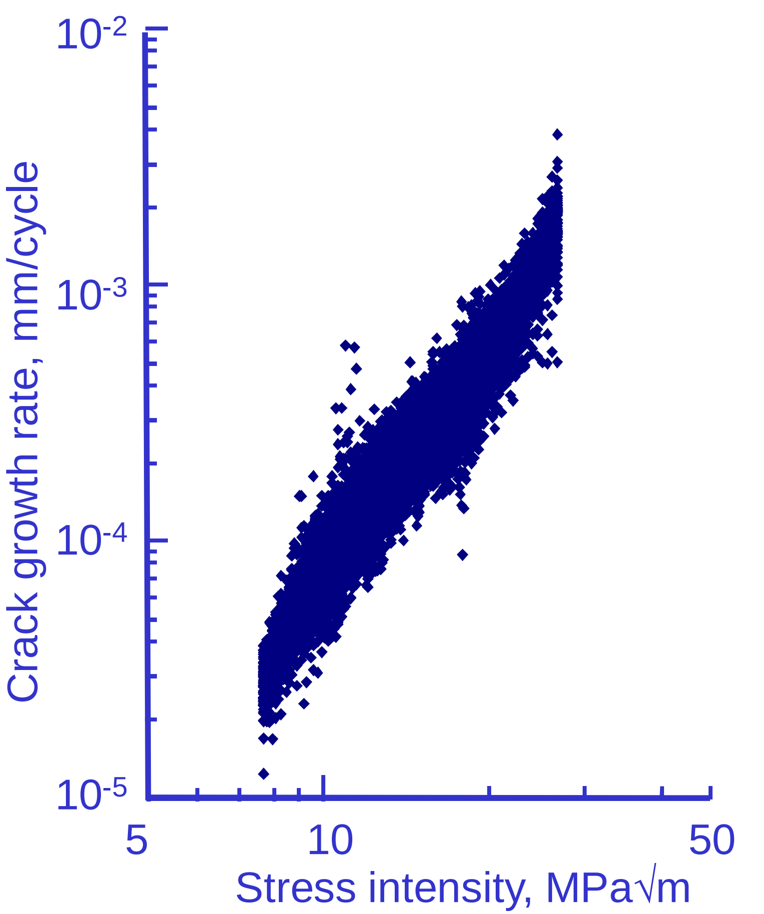


Virkler, Hillberry and Goel, "The Statistical Nature of Fatigue Crack Propagation", Journal of Engineering Materials and Technology, Vol. 101, 1979, 148-153

Crack Growth Rate Data



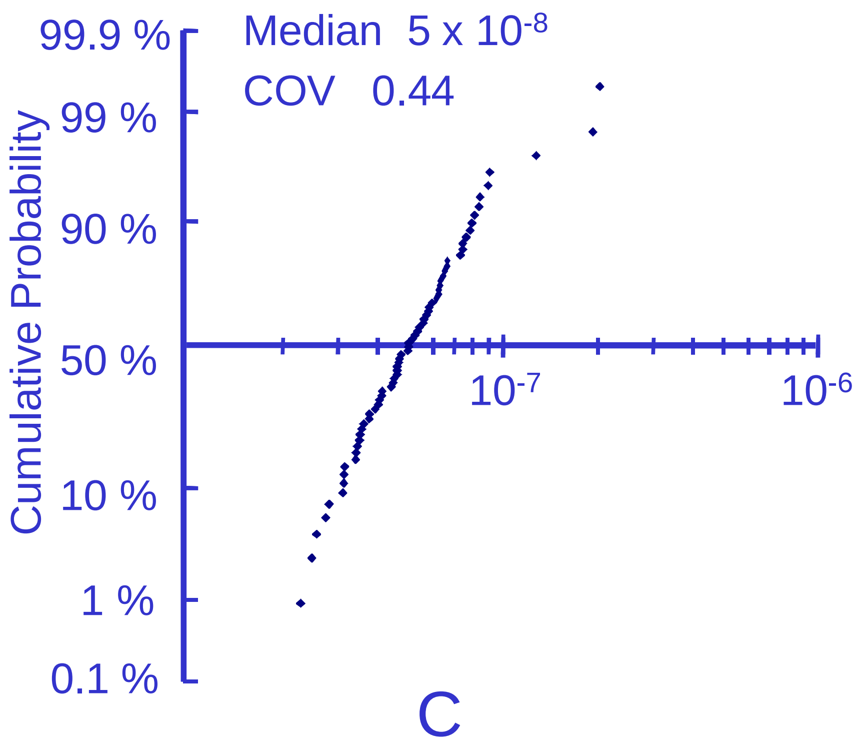
Fatigue Lives

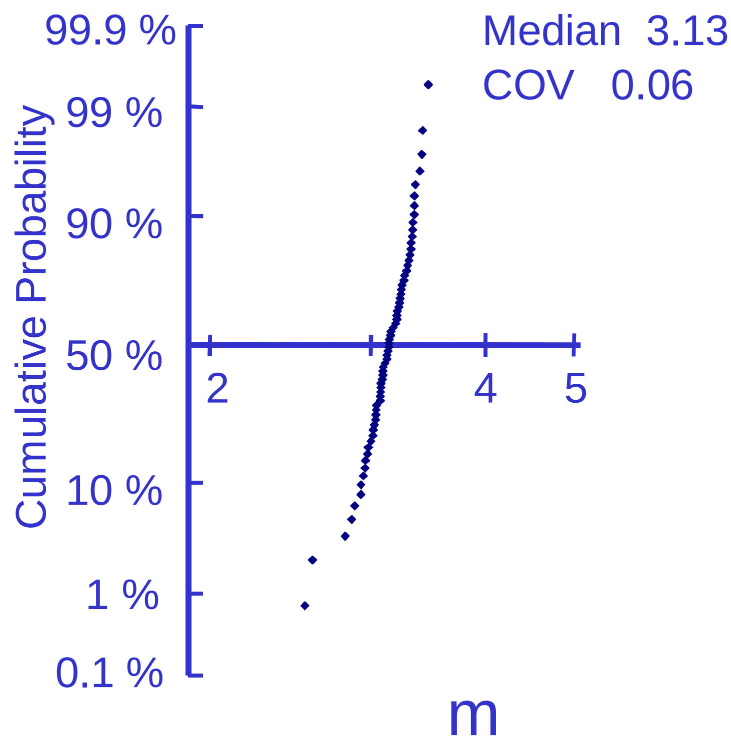


Crack Growth Rate

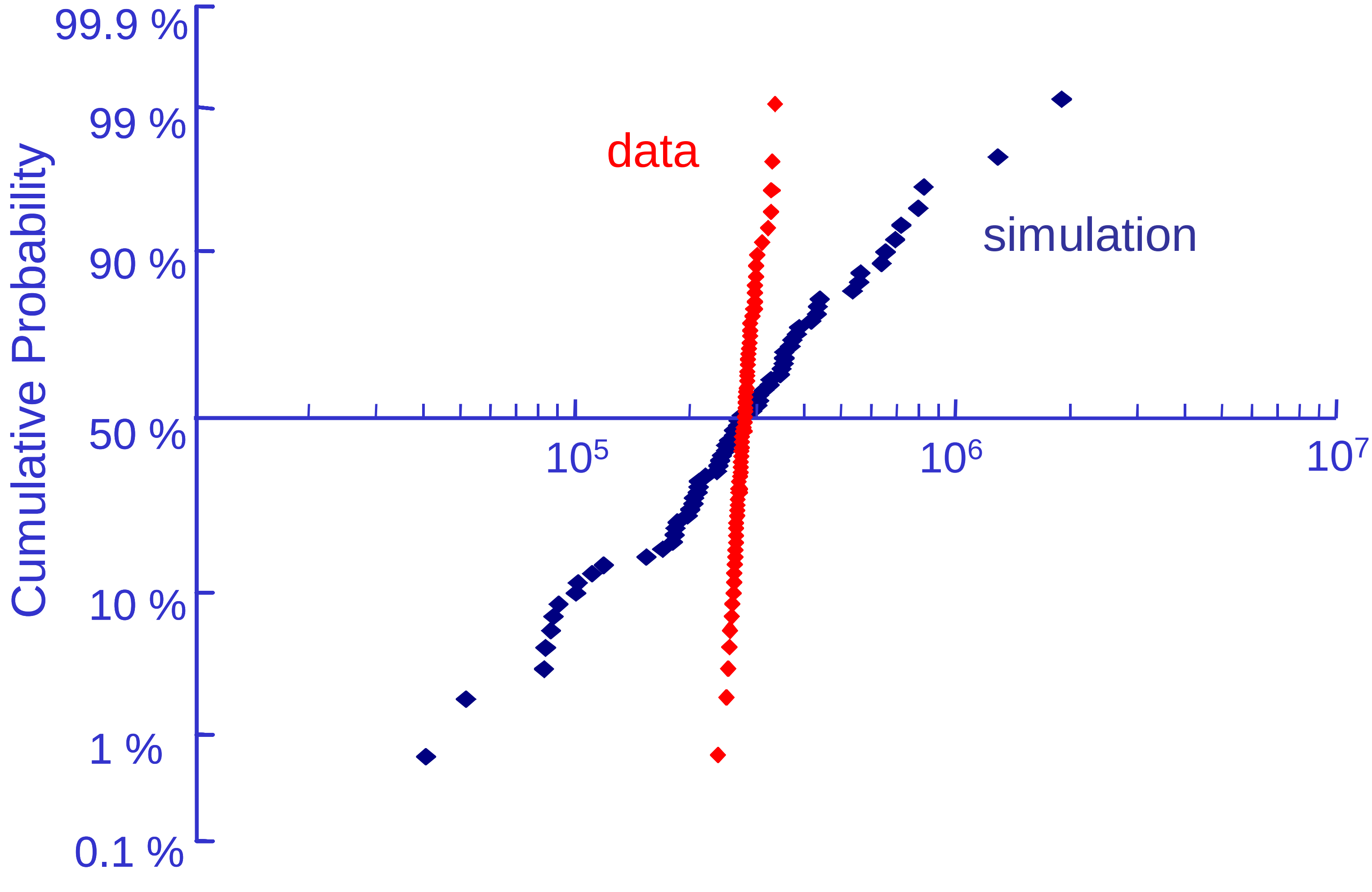
Crack Growth Properties

$$\frac{da}{dN} = C\,\Delta K^{m}$$





Simulated Data



Beware of Correlated Variables

$$N_f = \frac{a_f^{1-m/2} - a_i^{1-m/2}}{C \Delta S^m \pi^{\frac{m}{2}} \left(1-m/2\right)}$$

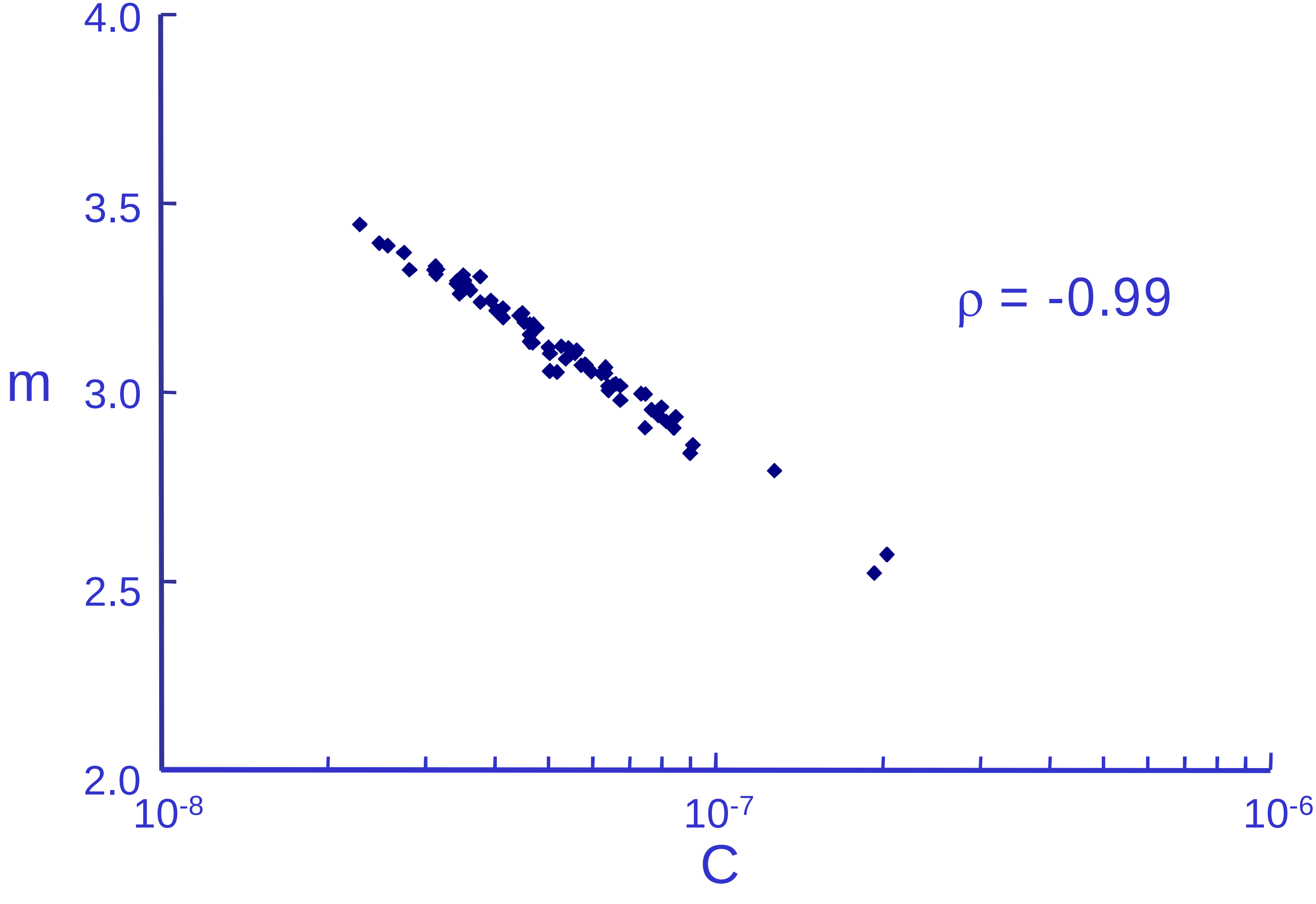
N_f and C are linearly related and should have the same variability, but

$$COV_{N_f} = 0.07$$

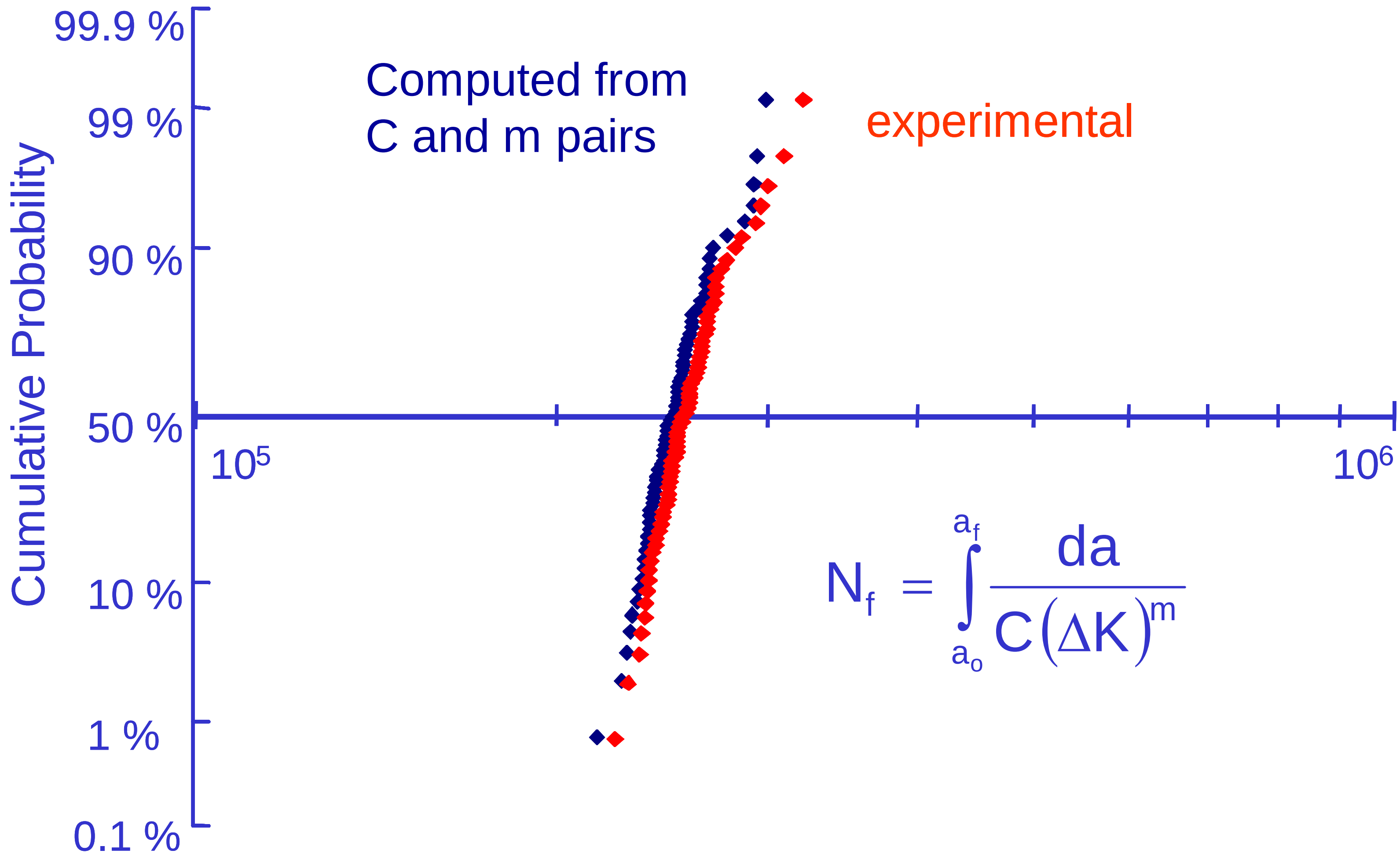
$$COV_C = 0.44$$

because C and m are correlated.

Correlation



Calculated Lives

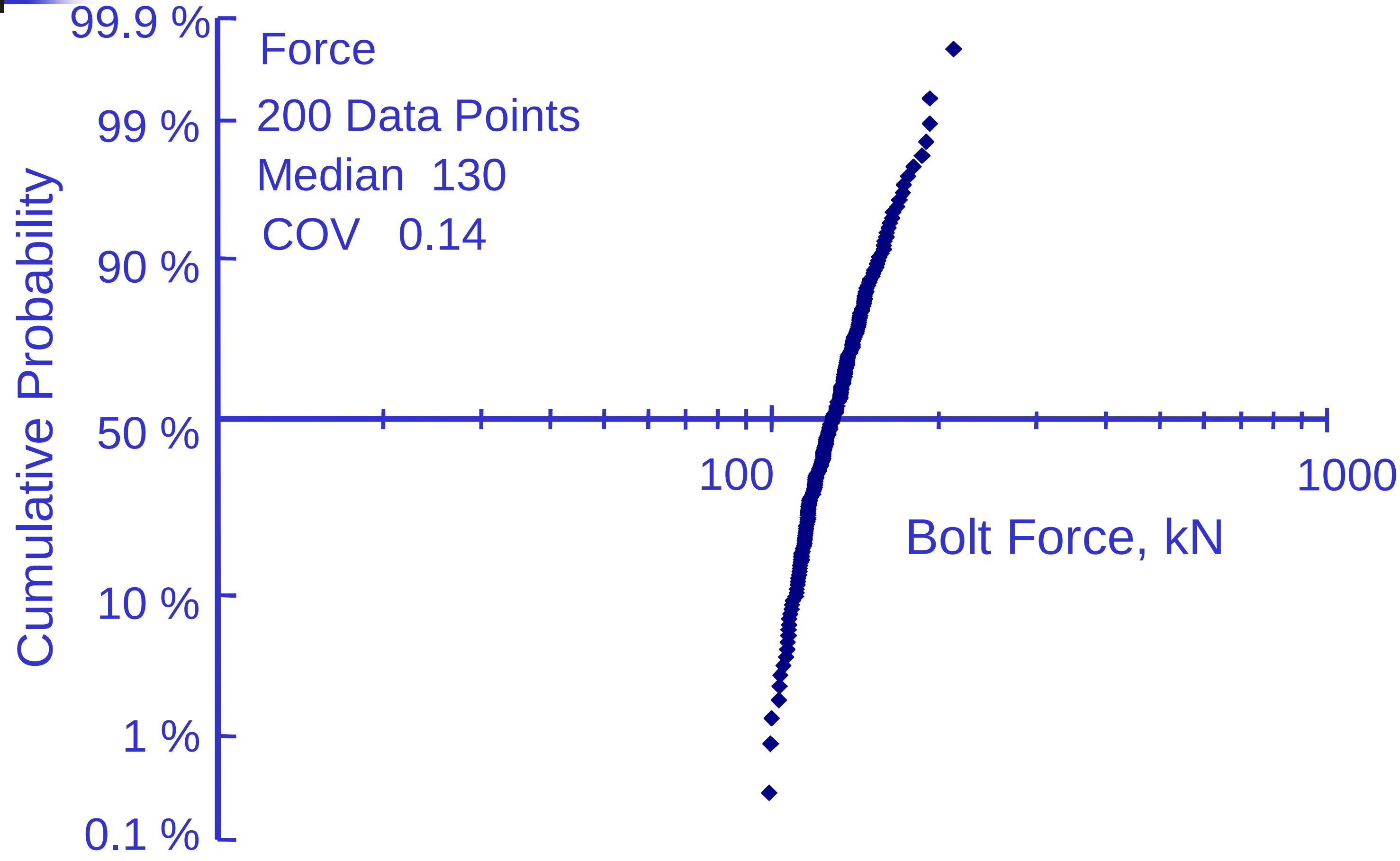


$$N_f = \int_{a_o}^{a_f} \frac{da}{C\left(\Delta K\right)^m}$$

Manufacturing/Processing Variability

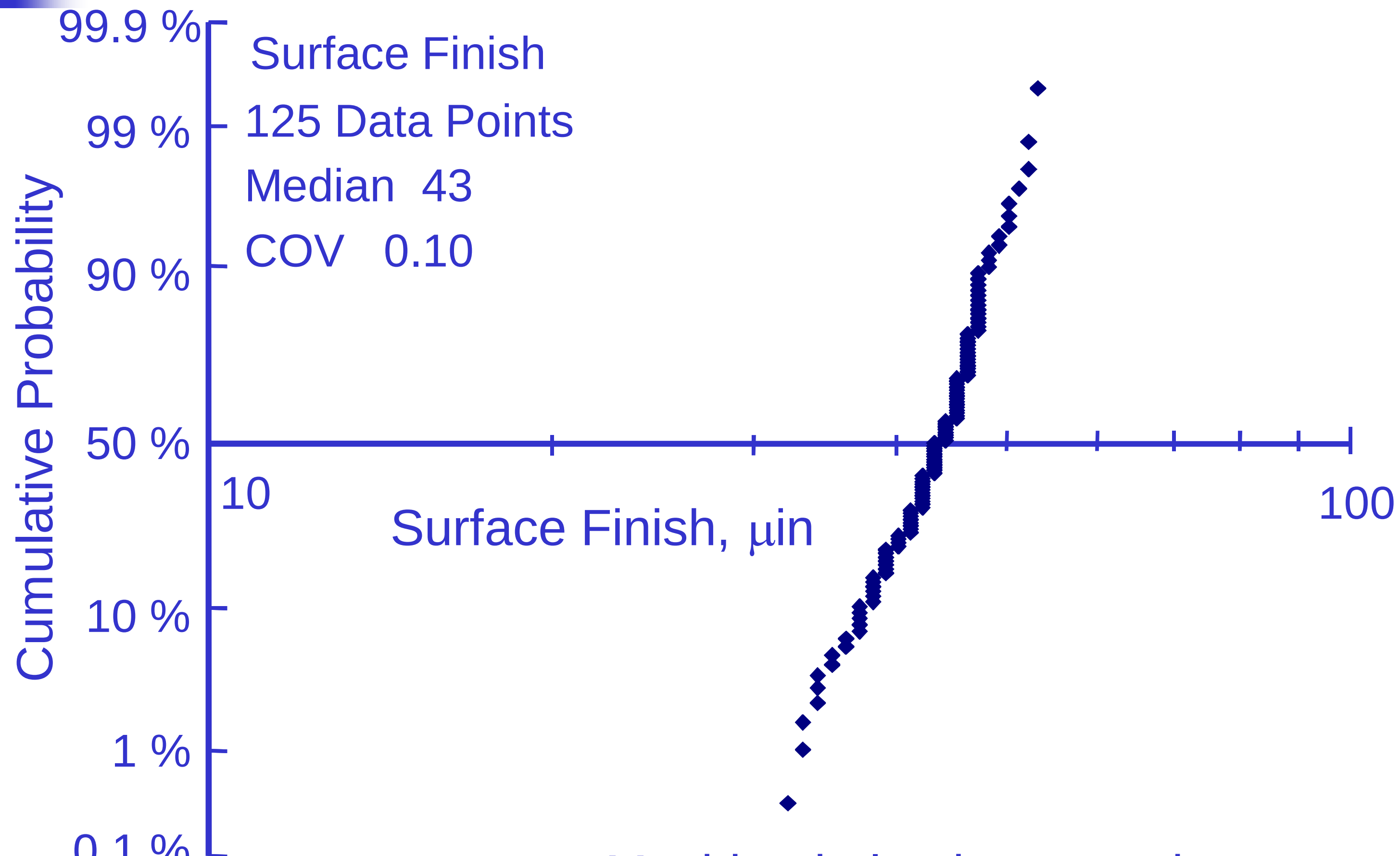
- Bolt Forces
- Surface Finish
- Drilled Holes

Variability in Bolt Force



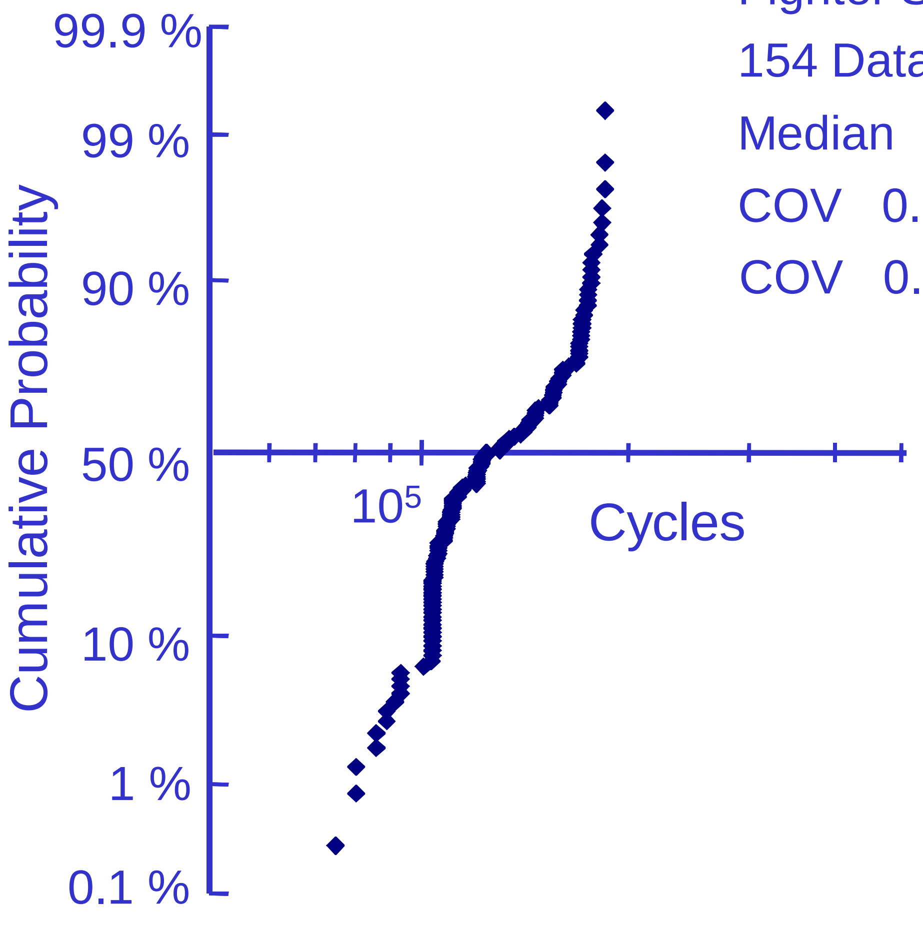
Preload force in bolts tightened to 350 Nm

Surface Roughness Variability



Machined aluminum casting

Drilled Holes



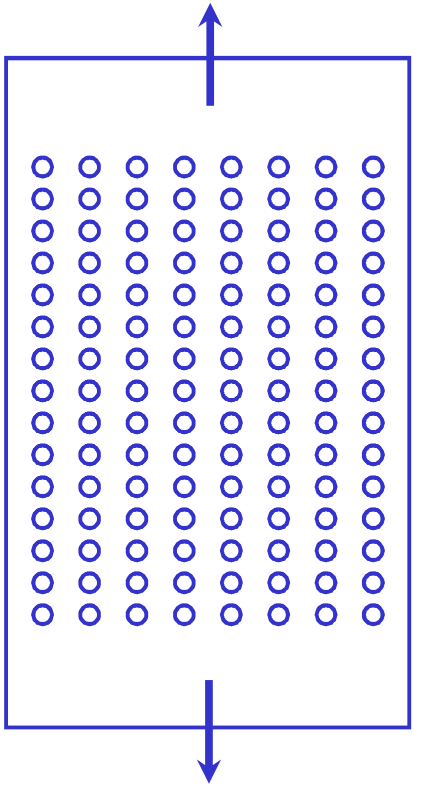
Fighter Spectrum

154 Data Points

Median 126,750

COV 0.22 in life

COV 0.07 in strength



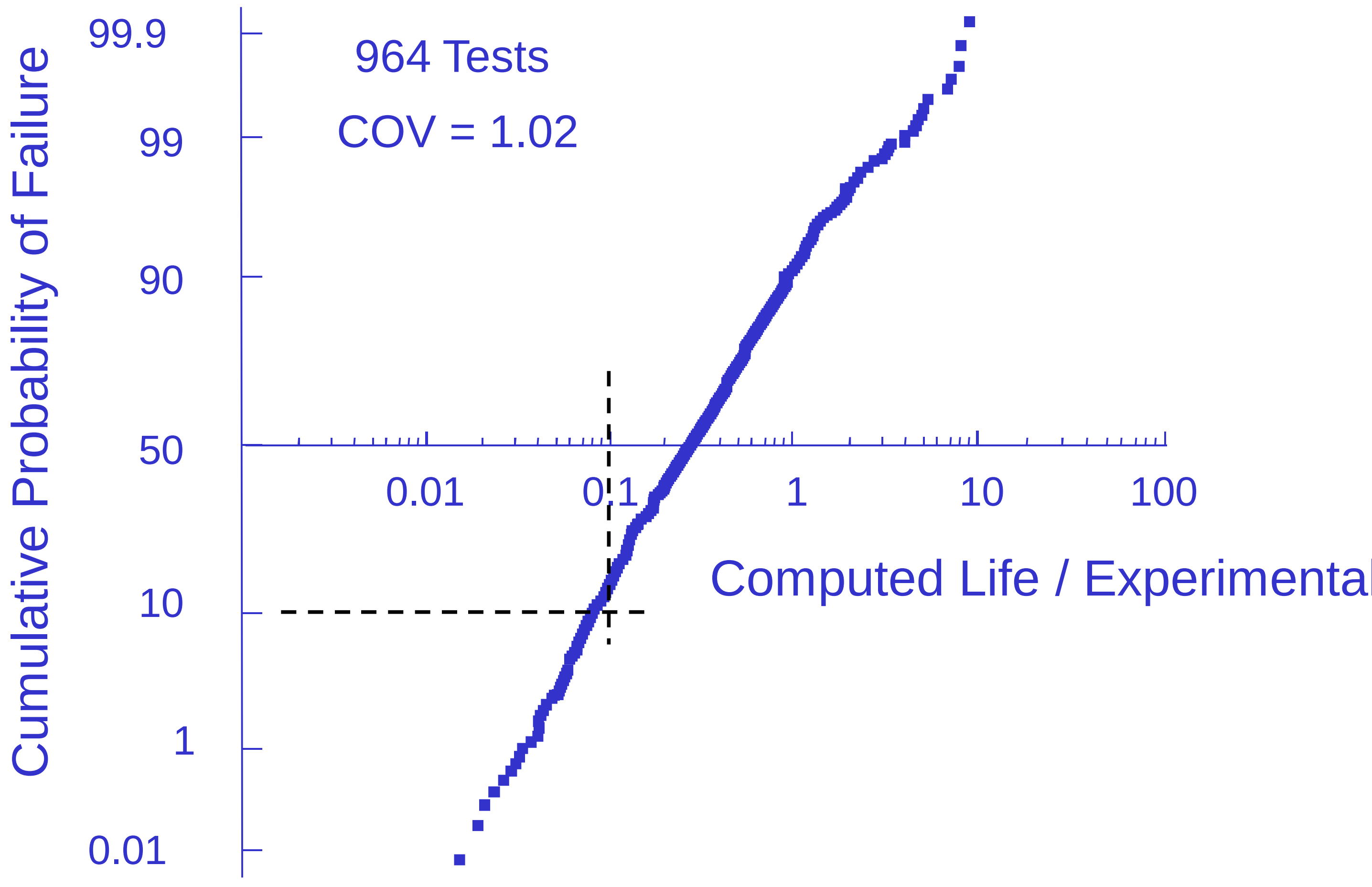
180 drilled holes in a single plate

From: J.P. Butler and D.A. Rees, "Development of Statistical Fatigue Failure Characteristics of 0.125-inch 2024-T3 Aluminum Under Simulated Flight-by-Flight Loading," ADA-002310 (NTIS no.), July 1974.

Analysis Uncertainty

- Miners Linear Damage rule
- Strain Life Analysis

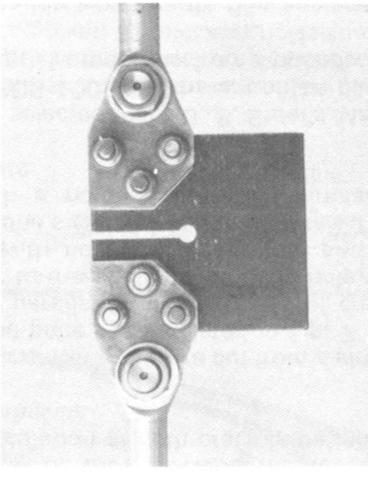
Miners Rule

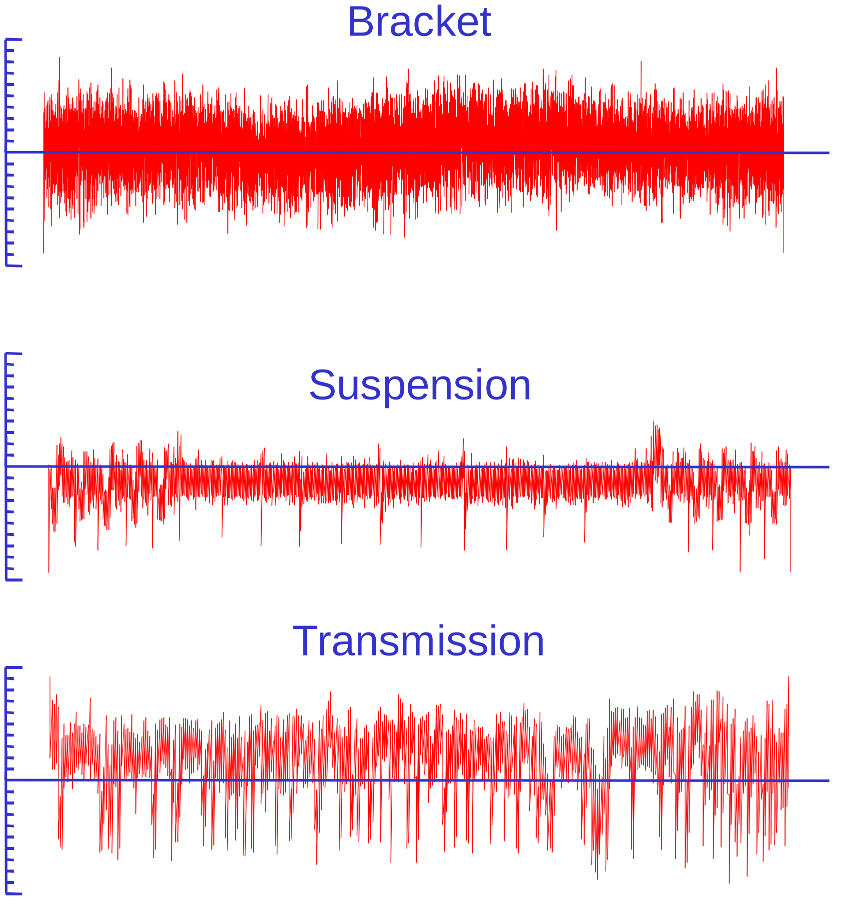


A safety factor of 10 in life would result in a 10% chance of failure

From Erwin Haibach “Betriebsfestigkeit”, Springer-Verlag, 2002

SAE Specimen

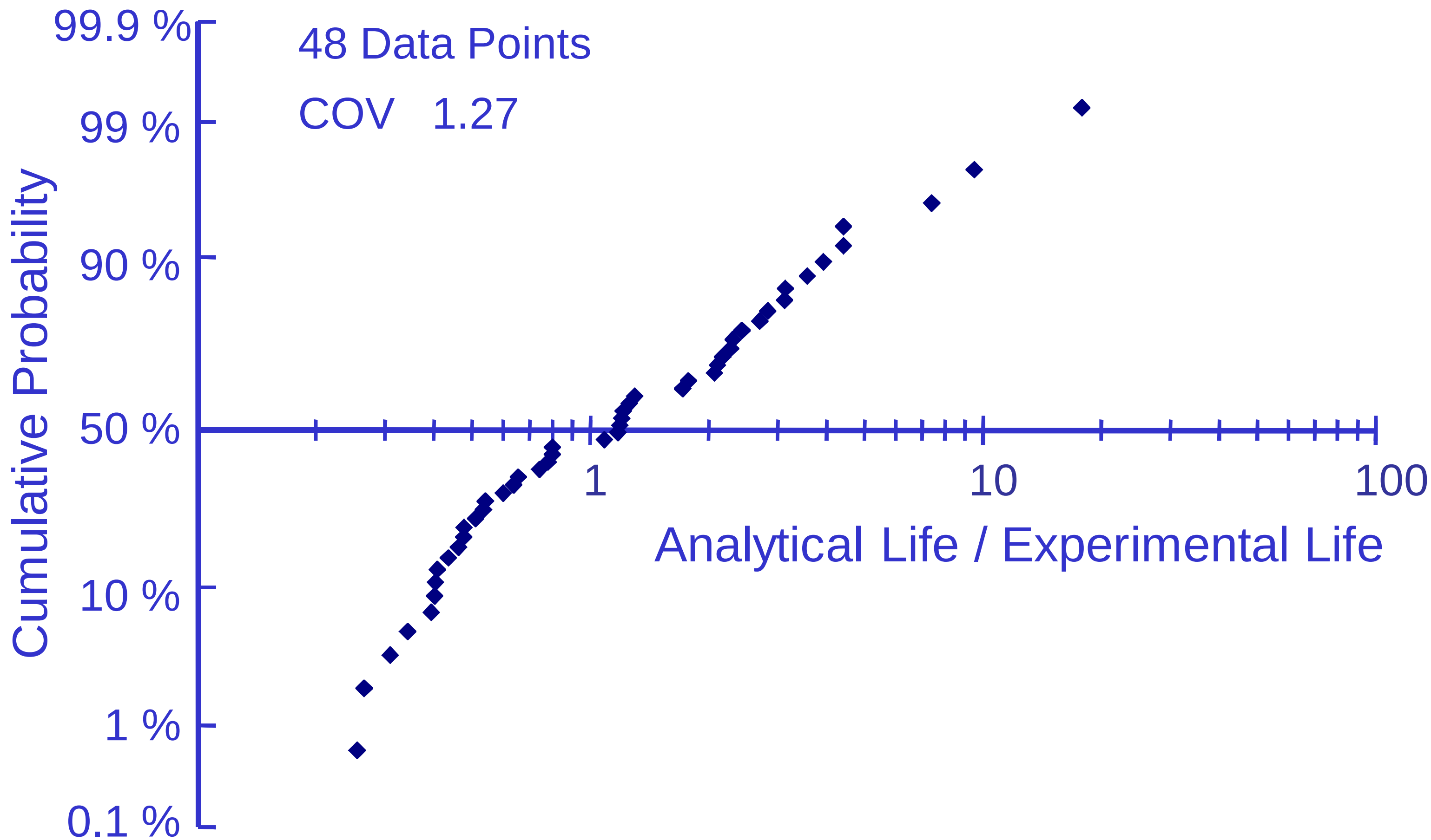




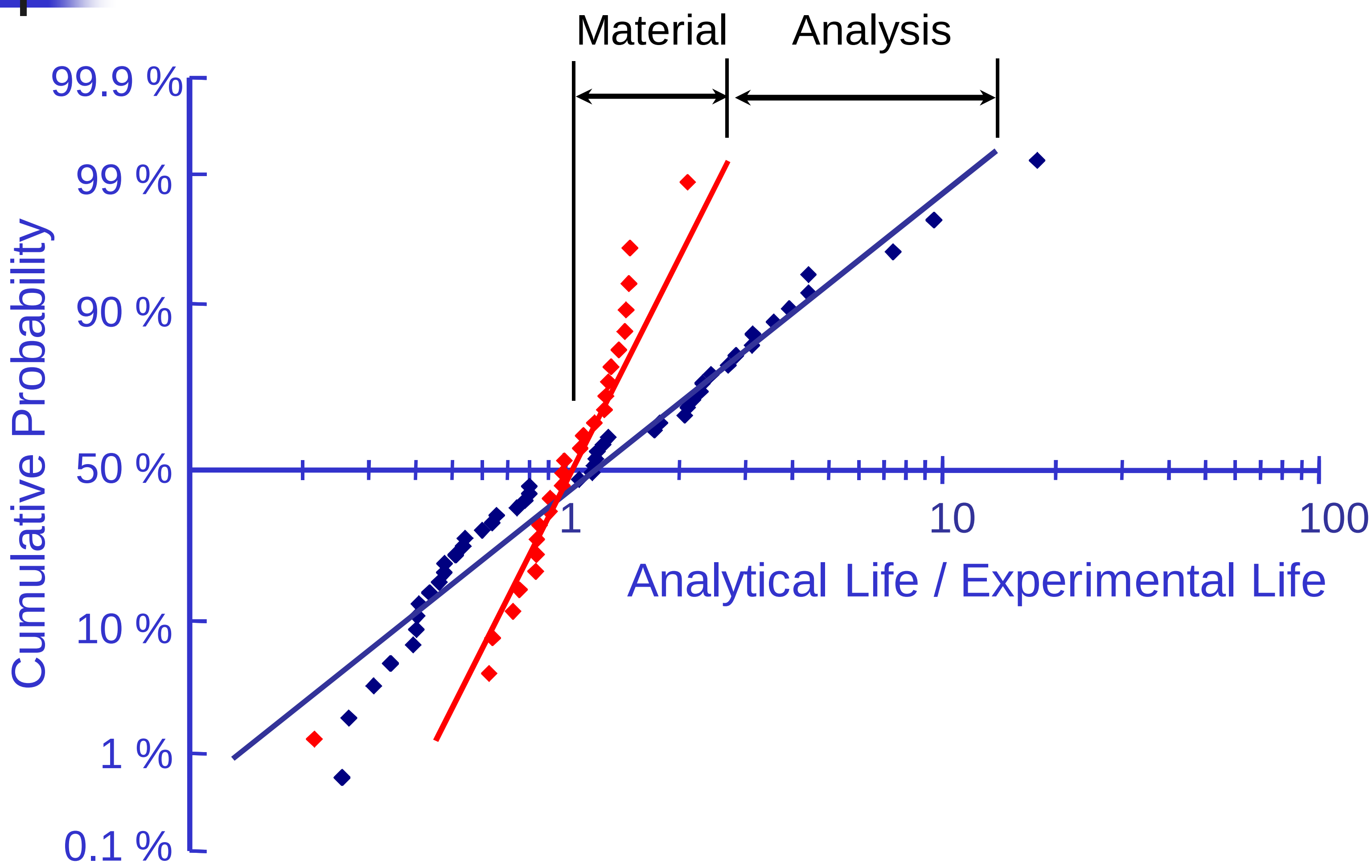
Fatigue Under Complex Loading: Analysis and Experiments, SAE AE6, 1977

Analysis Results

Strain-Life analysis of all test data



Material Variability



Strain-Life back calculation of specimen lives

Modeling Uncertainty

Analysis Uncertainty C_U = ?

The variability in reproducing the original strain life data from the material constants is $C_M \sim 0.44$

$$\text{COV} \quad C = \sqrt{\prod_{i=1}^{n} \left(1 + C_{X_i}{}^2\right)^{a_i^2} - 1}$$

$$1 + C_U{}^2 = \frac{1 + C_{N_f}{}^2}{1 + C_M{}^2}$$

$$C_U = 1.09$$

90% of the time the analysis is within a factor of 3 !

99% of the time the analysis is within a factor of 10 !

Variability from Multiple Sources

$$\text{COV} \quad C = \sqrt{\prod_{i=1}^{n}\left(1 + C_{X_i}{}^{2}\right)^{a_i{}^{2}} - 1}$$

Suppose we have 4 variables each with a COV = 0.1

The combined variability is COV = 0.29

Suppose we reduce the variability of one of the variables to 0.05

The combined variability is now COV = 0.27

If all of the COV's are the same, it doesn't do any good to reduce only one of them, you must reduce all of them !

Variability from Multiple Sources

$$\text{COV} \quad C = \sqrt{\prod_{i=1}^{n}\left(1 + C_{X_i}{}^{2}\right)^{a_i{}^{2}} - 1}$$

Suppose we have 3 variables each with a COV = 0.1 and one with COV = 0.4

The combined variability is COV = 0.65

Suppose we reduce the variability of these variables to 0.05

The combined variability is now COV = 0.60

If one of the COV's is large, it doesn't do any good to reduce the others, you must reduce the largest one !

Probabilistic Aspects of Fatigue